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STATISTICAL THEORY AND METHOD

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Short Title:
Statistical Theory Abstracts

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INTERNATIONAL JOURNAL OF ABSTRACTS
STATISTICAL THEORY AND METHOD

COVERAGE OF JOURNAL

THE aim of this journal of abstracts is to give complete coverage of papers in the field of statistical theory and new contributions to statistical method. Papers which report only applications or examples of existing statistical theory and method will not be included. There are approximately two hundred and fifty journals published in various parts of the world which are wholly or partly devoted to the field of statistical theory and method and which will be brought within the scope of this journal of abstracts. In due course it will be possible to issue a complete list of these journals. In the case of the following journals, however, being those which are wholly devoted to statistical theory—all contributions, whether a paper, note or miscellanea, will be abstracted :

Annals of Mathematical Statistics,
Biometrika
Journal, Royal Statistical Society (Series B)
Bulletin of Mathematical Statistics
Annals, Institute of Statistical Mathematics

Within the larger group of journals, which are not wholly devoted to statistical theory and method, there are some journals which have the vast majority of their contributions in this field. These journals, therefore, will be abstracted on a virtually complete basis :

Biometrics
Metrika
Metron
Review, International Statistical Institute
Technometrics
Sankhyā

After experience in the publication of this journal of abstracts it may be found desirable to add further journals to these lists. In any case readers may be assured that all papers properly to be included in this journal of abstracts will be included irrespective of such notification. If any reader of this journal discovers a paper which happens to have been overlooked, the General Editor will be pleased to be informed so that the appropriate abstract can be made : always provided that the date of publication is after 1st October 1958, when the abstracting for this journal commenced.

In addition to the ordinary journals, there are two kinds of publication which fall within the scope of this journal of abstracts. They are the experiment and other research station reports—which occur particularly in the North American region—and the reports of conferences, symposia and seminars. Whilst these latter may be included in the book review sections of journals it is unusual for any individual contribution to be noted at any length. These publications are, in effect, special collections of papers and for this reason the appropriate arrangements will be made for them to be included in this journal. By the same token, abstracts of papers given at conferences and reproduced in an appropriate journal will be disregarded until the definitive publication is available.

FORMAT OF JOURNAL

The abstracts will be up to 400 words long—the recommendation of UNESCO for the " long " abstract service : they will be in the English language although the language of the original paper will be indicated on the abstract together with the name of the abstractor. In addition, the address of the author(s) will be given in sufficient detail to facilitate contact in order to obtain further detail or request an offprint. Suitable indexes will be provided annually in a supplement.

The scheme of classification has been developed upon lines that will facilitate the transfer to punched cards of the code numbers allocated to each abstract: to allow for future development it is suggested that use is made of 4-column fields. Each abstract will have two classification numbers: the primary number in heavy type to indicate the basic topic of the paper and the secondary number in brackets to take account of the most important cross-reference. The sheets of the journal are colour-coded according to the twelve main sections of the classification and it should be noted that it is the main section number of the primary classification which determines the colour code for each abstract. It is believed that this method of colour-coding the pages will provide a distinctive visual aid to the identification of abstracts both when the journal is in bound form or dismantled and filed on cards or in binders. The format and simple binding allows of the following alternative treatments by users of the journal:

(*a*) Leave intact as a shelf-periodical.

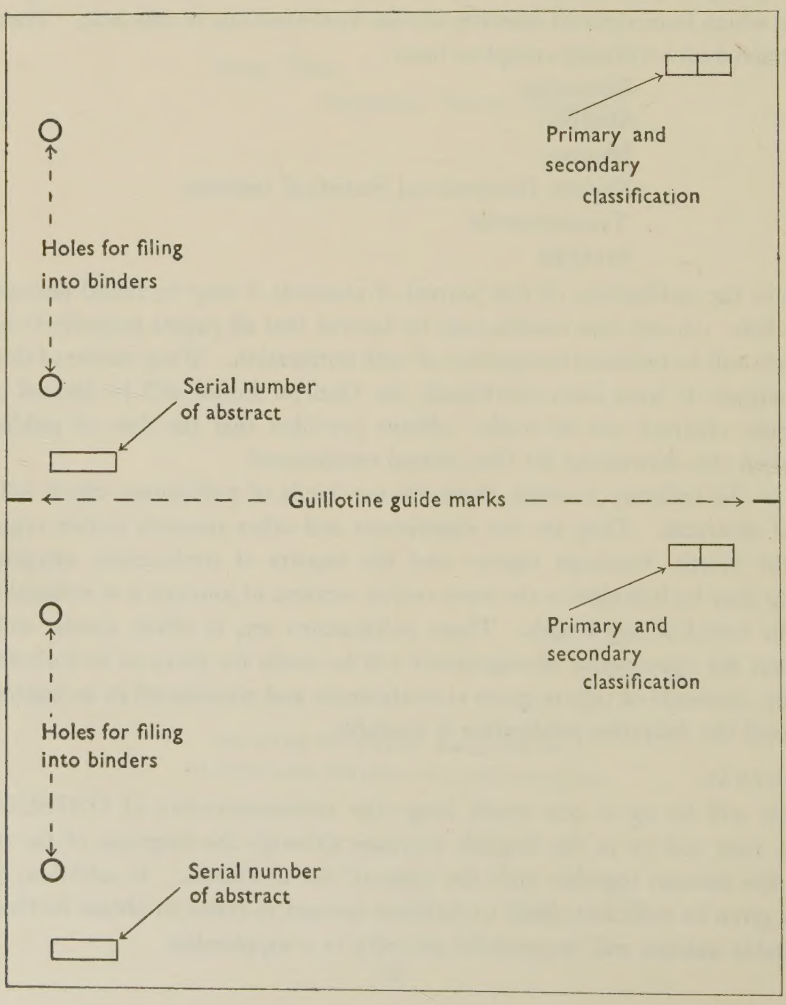
(*b*) Split and filed in page form according to the main sections of the classification.

(*c*) Split and guillotine (single cut) each page ready for:

(i) pasting onto standard index cards.

(ii) filing in loose-leaf or other binders—for which the appropriate holes are punched.

It will also be possible for those users who need a completely referenced file to insert skeleton cards or sheets according to the secondary classification number provided on each abstract.


Primary and secondary classification
Holes for filing into binders
Serial number of abstract
Guillotine guide marks
Primary and secondary classification
Holes for filing into binders
Serial number of abstract

INTERNATIONAL JOURNAL OF ABSTRACTS STATISTICAL THEORY AND METHOD

LIST OF PUBLICATIONS FOR REGULAR SCANNING

The following list of publications in the statistical and allied fields will be examined regularly for papers which should be abstracted for this Journal. Other journals will also be scrutinised from time to time and a check maintained by scanning other relevant abstracting journals. In addition, abstracts will be included of special collections of papers as published in reports of conferences, symposia and seminars together with the published reports of experiment and other research stations. The coverage of this Journal amounts to at least 400 sources of publication

Acta Acad. Cient., Lima	Acta de la Academia Cientifica, Lima	Peru
Acta Math., Stockh.	Acta Mathematica	Sweden
Acta Math. Acad. Sci. Hung.	Acta Mathematica Academiae Scientiarum Hungaricae	Hungary
Acta Math. Sinica	Acta Mathematica Sinica	China
Acta Pontif. Acad. Sci.	Acta Pontificiae Academiae Scientiarum	Italy
Advanced Management	Advanced Management	U.S.A.
Advanc. Enzymol.	Advances in Enzymology and Related Subjects	U.S.A.
Advanc. Protochem.	Advances in Protochemistry	U.S.A.
Advanc. Vet. Sci.	Advances in Veterinary Science	U.S.A.
Agric. Engng., St Joseph, Mich.	Agricultural Engineering	U.S.A.
Agriculture, Lond.	Agriculture (Min. of Agric.)	Great Britain
Agron. J.	Agronomy Journal	U.S.A.
Allg. Statist. Arch.	Allgemeines Statistisches Archiv	Germany
Amer. Econ. Rev.	American Economic Review	U.S.A.
Amer. Heart J.	American Heart Journal	U.S.A.
Amer. J. Clin. Nutr.	American Journal of Clinical Nutrition	U.S.A.
Amer. J. Clin. Path.	American Journal of Clinical Pathology	U.S.A.
Amer. J. Digestive Diseases	American Journal of Digestive Diseases	U.S.A.
Amer. J. Hum. Genet.	American Journal of Human Genetics	U.S.A.
Amer. J. Hyg.	American Journal of Hygiene	U.S.A.
Amer. J. Med.	American Journal of Medicine	U.S.A.
Amer. J. Physiol.	American Journal of Physiology	U.S.A.
Amer. J. Publ. Health	American Journal of Public Health	U.S.A.
Amer. J. Sci.	American Journal of Science	U.S.A.
Amer. J. Sociol.	American Journal of Sociology	U.S.A.
Amer. J. Vet. Res.	American Journal of Veterinary Research	U.S.A.
Amer. Nat.	American Naturalist	U.S.A.
Amer. Statistician	American Statistician	U.S.A.
Analyt. Chem.	Analytical Chemistry	U.S.A.
An. Soc. Cient. Argent.	Anales de la Sociedad Cientifica Argentina	Argentina
Ann. Acad. Sci. Fenn.	Annales Academiae Scientiarum Fennicae	Finland
Ann. Appl. Biol.	Annals of Applied Biology	Great Britain
Ann. Fac. Econ. Com. Catania	Annali della Facoltà di Economia e Commercio, Catania (University)	Italy
Ann. Fac. Econ. Com. Palermo	Annali della Facoltà di Economia e Commercio, Palermo (University)	Italy
Ann. Fac. Sci. Porto	Annaes da Faculdade de Ciências do Porto	Portugal
Ann. Hum. Genet.	Annals of Human Genetics (London)	Great Britain
Ann. Inst. Poincaré	Annales de l'Institut Henri Poincaré	France
Ann. Inst. Statist. Math., Tokyo	Annals of the Institute of Statistical Mathematics	Japan
Ann. Inst. Sup. Sci. Econ. Financ.	Annaes do Instituto Superior de Ciências Economicas e Financieras	Portugal
Ann. Ist. Statist. Bari	Annali dell'Istituto di Statistica, Bari (University)	Italy
Ann. Math. Statist.	Annals of Mathematical Statistics	U.S.A.
Ann. N.Y. Acad. Sci.	Annals of the New York Academy of Science	U.S.A.
Ann. Sci. Éc. Norm. Sup., Paris	Annales Scièntifiques de l'École Normale Supérieure	France
Ann. Sci. Écon. Appl.	Annales de Sciences Économique Appliquées	France
Ann. Soc. Sci. Bruxelles, I	Annales de la Société Scientifique de Bruxelles—Serie 1	Belgium

Ann. Statist., Rome	Annali di Statistica (Istituto Centrale di Statistica, Rome)	Italy
Ann. Univ. Lyon, A	Annales de l'Université de Lyon—Section A	France
Ann. Univ. Sci. Budapest. Sect. Math.	Annales Universitatis Scientiarum Budapestiensis de Rolando Eötvös nominatae—Sectio Mathematica	Hungary
Annu. Rev. Physiol.	Annual Review of Physiology	U.S.A.
Aplik. Mat.	Aplikace Matematiky	Czechoslovakia
Appl. Sci. Res., Hague	Applied Scientific Research, The Hague	Netherlands
Appl. Statist.	Applied Statistics	Great Britain
Archimede	Archimede	Italy
Arch. Biochem. Biophys.	Archives of Biochemistry and Biophysics	U.S.A.
Arch. Math., Karlsruhe	Archiv der Mathematik	Germany
Ark. Mat.	Arkiv för Matematik	Sweden
Atti (mem.) Accad. Lincei	Atti della Accademia Nazionale dei Lincei	Italy
Atti Riun. Sci., Soc. Ital. Statist.	Atti della Riunione Scientifica, Società Italiana di Statistica, Rome	Italy
Aust. J. Agric. Res.	Australian Journal of Agricultural Research	Australia
Aust. J. Appl. Sci.	Australian Journal of Applied Science	Australia
Aust. J. Biol. Sci.	Australian Journal of Biological Sciences	Australia
Aust. J. Phys.	Australian Journal of Physics	Australia
Aust. J. Sci.	Australian Journal of Science	Australia
Aust. J. Statist.	Australian Journal of Statistics	Australia
Beil. Mber. Öst. Inst. Wirtschaftsf.	Beilagen zu den Monatsberichten des Österreichischen Institutes für Wirtschaftsforschung	Austria
Bell Lab. Record	Bell Laboratories Record	U.S.A.
Bell Syst. Monographs	Bell System Monographs	U.S.A.
Bell Syst. Tech. J.	Bell System Technical Journal	U.S.A.
Biometrics	Biometrics	U.S.A.
Biometrika	Biometrika	Great Britain
Biom. Zeit.	Biometrische Zeitschrift	Germany
Blä. Dtsch. Ges. Versich.-math.	Blätter der Deutschen Gesellschaft für Versicherungsmathematik	Germany
Bol Fac. Filos. Ciênt., S. Paulo	Boletim da Faculdade de Filosofía, Ciências e Letras, São Paulo	Brazil
Bol. Inst. Actuar. Port.	Boletim do Instituto dos Actuarios Portugueses	Portugal
Bol. Soc. Mat. Mexicana	Boletim de la Sociedad Matemática Mexicana	Mexico
Boll. Cent. Ric. Operat.	Bollettino del Centro per la Ricerca Operativa	Italy
Boll. Un. Mat. Ital.	Bollettino dell'Unione Matematica Italiana	Italy
Brit. J. Nutr.	British Journal of Nutrition	Great Britain
Brit. J. Prev. Soc. Med.	British Journal of Preventive and Social Medicine	Great Britain
Brit. J. Sociol.	British Journal of Sociology	Great Britain
Brit. J. Statist. Psychol.	British Journal of Statistical Psychology	Great Britain
Bull. Acad. Polon. Sci. III	Bulletin de l'Académie des Sciences (Classe 3)	Poland
Bull. Amer. Soc. Test. Mat.	Bulletin, American Society for Testing Materials	U.S.A.
Bull. Ass. Actuair. Suisses	Bulletin de l'Association des Actuaires Suisses	Switzerland
Bull. Calcutta Math. Soc.	Bulletin of the Calcutta Mathematical Society	India
Bull. Calcutta Statist. Ass.	Bulletin, Calcutta Statistical Association	India
Bull., C.S.I.R.O., Aust.	Bulletins, Commonwealth Scientific and Industrial Research Organisation	Australia
Bull. Indian Soc. Qual. Contr.	Bulletin, Indian Society of Quality Control	India
Bull. Indian Stand. Inst.	Bulletin, Indian Standards Institution	India
Bull. Inst. Égypte	Bulletin de l'Institut d'Égypte, Cairo	Egypt
Bull. Jap. Statist. Soc.	Bulletin of the Japan Statistical Society	Japan
Bull. Math. Statist.	Bulletin of Mathematical Statistics, Research Association of Statistical Sciences	Japan
Bull. Nat. Inst. Sci. India	Bulletin, National Institute of Sciences of India	India
Bull. Oxf. Univ. Inst. Statist.	Bulletin of the Oxford University Institute of Statistics	Great Britain
Bull. Res. Inst. Kerala	Bulletin, Central Research Institute, Kerala (University)	India
Bull. Sci. Math.	Bulletin des Sciences Mathématique	France
Bull. Soc. Math. Belg.	Bulletin de la Société Mathématique de Belgique, Gembloux	Belgium

Bull. Soc. Math. France	Bulletin de la Société Mathématique de France	France
Bull. Statist. Soc., N.S.W.	Bulletin of the Statistical Society, N.S. Wales	Australia
Canad. J. Econ. Polit. Sci.	Canadian Journal of Economic and Political Science	Canada
Cancer, New York	Cancer (New York)	U.S.A.
Cas. pěst. math.	Casopis pro pěstování mathematiky	Czechoslovakia
Chem. Engng. Progr.	Chemical Engineering Progress	U.S.A.
Clin. Chem.	Clinical Chemistry	U.S.A.
Colect. Estud.	Colectanea de Estudos	Spain
Coll. Mat.	Collectanea Matematica	Spain
Colloq. Math.	Colloquium Mathematicum	Poland
Comment. Math. Helvetia	Commentarii Mathematica Helvetia	Switzerland
Comment. Pontif. Acad. Sci.	Commentationes Pontificiae Academiae Scientiarum	Italy
Compos. Math., Groningen	Compositio Mathematica, Groningen	Netherlands
C.R. Acad. Sci., Paris	Comptes Rendus de l'Académie des Sciences, Paris	France
Cornell Vet.	Cornell Veterinarian	U.S.A.
Current Science	Current Science	India
Curso Mira Fernandes	Curso Mira Fernandes	Portugal
Czech. Math. J.	Czechoslowak Mathematical Journal	Czechoslovakia
Dacca Univ. Statist. Bull.	Dacca University Statistical Bulletin	Pakistan
Defense Sci. J.	Defense Science Journal	India
Dokl. Akad. Nauk. SSSR	Doklady Akademiia Nauk, SSSR.	U.S.S.R.
Dopov. Akad. Nauk. Ukrain, SSR	Dopovedy Akademiia Nauk Ukrainskoj, SSR	U.S.S.R.
Écon. Appl.	Économie Appliquée	France
Econ. Internazionale	Economia Internazionale	Italy
Econ. J.	Economic Journal	Great Britain
Écon. Leban. Arabe	L'Économie Lebanaise & Arabe, Beirut	Lebanon
Econ. Record	Economic Record	Australia
Econometrica	Econometrica	U.S.A.
Economica	Economica	Great Britain
Edinb. Math. Notes	Edinburgh Mathematical Notes	Great Britain
Égypte Contemp.	L'Égypte Contemporaine, Cairo	Egypt
Egyptian Statist. J.	Egyptian Statistical Journal	Egypt
Ekon. Tidskr.	Ekonomisk Tidskrifts	Sweden
Emp. J. Exp. Agric.	Empire Journal of Experimental Agriculture	Great Britain
Endocrinology	Endocrinology	U.S.A.
Engineer, Lond.	Engineer	Great Britain
Engineering, Lond.	Engineering	Great Britain
Ergebn. Naturw. Grenzgeb.	Ergebnisse der exakten Naturwissenschaften	Germany
Esta	Esta	Spain
Estadistica	Estadistica	U.S.A.
Euclides, Groningen	Euclides, Groningen	Netherlands
Eugen. Rev.	Eugenics Review	Great Britain
Evolution	Evolution	U.S.A.
Food Res.	Food Research	U.S.A.
Food Tech.	Food Technology	U.S.A.
Ganita	Ganita, Lucknow-Allahabad (Universities)	India
Gaz. Mat.	Gazeta de Matemática	Portugal
G. Economisti	Giornale degli Economisti e Annali di Economia	Italy
G. Ist. Ital. Attuari	Giornale dell'Istituto Italiano degli Attuari	Italy
Genetics	Genetics	U.S.A.
Genus	Genus	Italy
Growth	Growth	U.S.A.
Harvard Busin. Rev.	Harvard Business Review	U.S.A.
Helvetica Phys. Acta	Helvetica Physica Acta	Switzerland

Heredity	Heredity	Great Britain
Hum. Biol.	Human Biology	U.S.A.
IFO-Studien	IFO-Studien	Germany
Inc. Statist.	Incorporated Statistician	Great Britain
Indag. Math.	Indagationes Mathematicae, Amsterdam	Netherlands
Indian Coun. Agric. Res. Bull.	Indian Council of Agricultural Research : Bulletin	India
Indian Coun. Agric. Res. Statist. Newsletter	Indian Council of Agricultural Research : Statistical Newsletter	India
Indian Econ. J.	Indian Economic Journal	India
Indian Econ. Rev.	Indian Economic Review	India
Indian J. Agric. Econ.	Indian Journal of Agricultural Economics	India
Indian J. Phys.	Indian Journal of Physics	India
Indian J. Power & River Devel.	Indian Journal of Power and River Development	India
Indian J. Soc. Work	Indian Journal of Social Work	India
Indian Math. Soc. J.	Indian Mathematical Society Journal	India
Industr. Engng. Chem.	Industrial and Engineering Chemistry	U.S.A.
Industr. Qual. Contr.	Industrial Quality Control	U.S.A.
Industria, Milano	L'Industria	Italy
Iowa Acad. Sci. J.	Iowa Academy of Science Journal	U.S.A.
Iowa Exp. Sta. Publ.	Iowa State College Experiment Station Publications	U.S.A.
Izv. Akad. Nauk, SSSR	Izvestiya Akademiia Nauk, SSSR	U.S.S.R.
Izv. Akad. Nauk, Uzbek., SSR.	Izvestiya Akademiia Nauk, Uzbekskoj, SSR.	U.S.S.R.
Jb. Nat.-Ökon. Statist.	Jahrbücher für Nationalökonomie und Statistik	Germany
Jber. Dtsch. Mathver.	Jahresbericht der Deutschen Mathematikervereinigung	Germany
J. Agric. Food Chem.	Journal of Agricultural and Food Chemistry	U.S.A.
J. Agric. Sci.	Journal of Agricultural Science	Great Britain
J. Amer. Pharm. Ass.	Journal of the American Pharmaceutical Association (Scientific Edition)	U.S.A.
J. Amer. Statist. Ass.	Journal, American Statistical Association	U.S.A.
J. Amer. Vet. Med. Ass.	Journal of American Veterinary Medicine Association	U.S.A.
J. Animal Sci.	Journal of Animal Sciences	U.S.A.
J. Appl. Physiol.	Journal of Applied Physiology	U.S.A.
J. Ass. Comp. Mach.	Journal of the Association of Computing Machinery	U.S.A.
J. Aust. Inst. Agric. Sci.	Journal of the Australian Institute of Agricultural Science	Australia
J. Aust. Math. Soc.	Journal of the Australian Mathematical Society	Australia
J. Biol. Chem.	Journal of Biological Chemistry	U.S.A.
J. Chronic Diseases	Journal of Chronic Diseases	U.S.A.
J. Clin. Invest.	Journal of Clinical Investigation	U.S.A.
J. Dairy Sci.	Journal of Dairy Science	U.S.A.
J. Ecology	Journal of Ecology	Great Britain
J. Econ. Ent.	Journal of Economic Entomology	U.S.A.
J. Exp. Educ.	Journal of Experimental Education	U.S.A.
J. Exp. Med.	Journal of Experimental Medicine	U.S.A.
J. Forestry	Journal of Forestry	U.S.A.
J. Gen. Physiol.	Journal of General Physiology	U.S.A.
J. Genet.	Journal of Genetics	India
J. Heredity	Journal of Heredity	U.S.A.
J. Hyg., Camb.	Journal of Hygiene, Cambridge	Great Britain
J. Industr. Econ.	Journal of Industrial Economics	Great Britain
J. Industr. Engng.	Journal of Industrial Engineering	U.S.A.
J. Indian Soc. Agric. Statist.	Journal of the Indian Society of Agricultural Statistics	India
J. Inst. Actuar.	Journal of the Institute of Actuaries	Great Britain
J. Inst. Actuar. Stud. Soc.	Journal of the Institute of Actuaries Students' Society	Great Britain
J. Madras Univ., B	Journal of Madras University, Section B.	Great Britain
J. Mammalogy	Journal of Mammalogy	U.S.A.
J. Marine Res.	Journal of Marine Research	U.S.A.
J. Math. Pures Appl.	Journal de Mathématiques Pures et Appliquées	France
J. Math. Soc. Japan	Journal of the Mathematical Society of Japan	Japan
J. Nutrition	Journal of Nutrition	U.S.A.
J. Operat. Res.	Journal of the Operations Research Society of America	U.S.A.

J. Pharmacol.	Journal of Pharmacology and Experimental Therapeutics	U.S.A.
J. Plant Physiol.	Journal of Plant Physiology	U.S.A.
J. Polit. Econ.	Journal of Political Economy	U.S.A.
J. reine angew. Math.	Journal für reine und angewandte Mathematik	Germany
J. Res., Nat. Bur. Stand.	Journal of Research, National Bureau of Standards, Washington	U.S.A.
J. Roy. Soc., N.S.W.	Journal of the Royal Society, N.S. Wales	Australia
J. R. Statist. Soc. A & B	Journal of the Royal Statistical Society (Series A & B)	Great Britain
J. Sci. Engng.	Journal of Science and Engineering	India
J. Sci. Industr. Res.	Journal of Scientific and Industrial Research	U.S.A.
J. Sci. Industr. Res., India	Journal of Scientific and Industrial Research	India
J. Soc. Statist. Paris	Journal de la Société Statistique de Paris	France
J. Statist. Social Inquiry Soc.	Journal of the Statistical and Social Inquiry Society of Ireland	Ireland
J. Univ. Baroda	Journal, Maharaja Sayajiroa, Baroda University	India
Kōdai, Tokyo Inst. Tech.	Kōdai Mathematical Seminar reports, Tokyo Institute of Technology	Japan
Magy. Tud. Akad. III *Oszt. Közl.*	Magyar Tudományos Akadémia Matematikai és Fizikai Osztályának Közleményei	Hungary
Management Sci.	Management Science	U.S.A.
Manager (India)	Manager	India
Manchester School	Manchester School	Great Britain
Matemática	Matemática, Notas del Instituto de Matemática Facultad de Ingeniería de Rosario	Argentine
Mat. Lapok	Matematikai Lapok	Hungary
Math. Ann.	Mathematische Annalen	Germany
Math. Gaz.	Mathematical Gazette	Great Britain
Math. Nachr.	Mathematische Nachrichten	Germany
Math. Stud.	Mathematics Students	India
Math. Tab., Wash.	Mathematical Tables and other aids to Computation	U.S.A.
Math., Tech., Wirtschaft	Mathematik, Technik, Wirtschaft (Zeitschrift für moderne Rechentechnik und Automation)	Austria
Math. Zeit.	Mathematische Zeitschrift	Germany
Meded. Landbhoogesch., Wageningen	Mededeelingen van de Landbouwhoogeschool, Wageningen	Netherlands
Mem. Coll. Sci., Kyoto	Memoirs of the College of Science, Kyoto (University)	Japan
Mem. Fac. Sci. A, *Kyusyu*	Memoirs of the Faculty of Science (Series A), Kyusyu (University)	Japan
Mém. Soc. Sci. Liége	Mémoires de la Société Royale des Sciences de Liége	Belgium
Metabolism	Metabolism	U.S.A.
Met. Mag., Lond.	Meteorological Magazine	Great Britain
Metrika	Metrika	Germany
Metroeconomica	Metroeconomica	Italy
Metron	Metron	Italy
Mitt. Ver. Schweiz. Versich.-Math.	Mitteilungen der Vereinigung Schweizerisher Versicherungs-Mathematiker	Switzerland
Monatsh. Math.	Monatshefte für Mathematik	Austria
Monthly Not. R. Astr. Soc.	Monthly Notices of the Royal Astronomical Society	Great Britain
Nat. Bur. Stand., Statist. Rep. & Publ.	National Bureau of Standards, Washington : Statistical Reports and Publications	U.S.A.
Nature	Nature (London)	Great Britain
Ned. Verzek. Actuar. Bij.	Nederlands Verzekeringsarchief, Actuarieel Bijvoegsel, The Hague	Netherlands
Nieuw Arch. Wisk.	Nieuw Archief voor Wiskunde, Groningen	Netherlands
Nieuw Tijdschr. Wisk.	Nieuw Tijdschrift voor Wiskunde, Groningen	Netherlands

Operat. Res. Quart.	Operational Research Quarterly	Great Britain
Osaka Math. J.	Osaka Mathematical Journal, Osaka (University)	Japan
Öst. Ingen.-arch.	Österreichisches Ingenieurarchiv	Austria
Oxf. Econ. Papers	Oxford Economic Papers	Great Britain
Pakistan Econ. J.	Pakistan Economic Journal	Pakistan
Phil. Mag.	Philosophical Magazine	Great Britain
Phil. Phenom. Res.	Philosophical and Phenomenological Research	Great Britain
Phil. Trans.	Philosophical Transactions of the Royal Society	Great Britain
Population Stud.	Population Studies	Great Britain
Portug. Math.	Portugaliae Mathematica	Portugal
Poultry Sci.	Poultry Science	U.S.A.
Proc. Amer. Soc. Hort. Sci.	Proceedings, American Society for Horticultural Science	U.S.A.
Proc. Amer. Soc. Qual. Contr.	Proceedings of the American Society for Quality Control	U.S.A.
Proc. Amer. Soc. Test. Mat.	Proceedings of the American Society for Testing Materials	U.S.A.
Proc. Camb. Phil. Soc.	Proceedings of the Cambridge Philosophical Society	Great Britain
Proc. Indian Acad. Sci.	Proceedings of the Indian Academy of Sciences	India
Proc. Indian Ass. Cult. Sci.	Proceedings of the Indian Association for Cultivation of Science	India
Proc. Indian Sci. Congr.	Proceedings of the Indian Science Congress	India
Proc. Inst. Statist. Math.	Proceedings of the Institute of Statistical Mathematics	Japan
Proc. Japan Acad.	Proceedings of the Japan Academy	Japan
Proc. Kon. Ned. Akad., Wetensch. A	Proceedings, Koninklijke Nederlandsche Akademie van Wetenschappen (A)	Netherlands
Proc. Lahore Phil. Soc.	Proceedings of the Lahore Philosophical Society	India
Proc. Lond. Math. Soc.	Proceedings of the London Mathematical Society	Great Britain
Proc. Math. Phys. Soc. Egypt	Proceedings of the Mathematical and Physical Society of Egypt	Egypt
Proc. Nat. Acad. Sci. India	Proceedings of the National Academy of Sciences	India
Proc. Nat. Inst. Sci. India (A & B)	Proceedings of the National Institute of Sciences of India (Section A & B)	India
Proc. Pakistan Statist. Ass.	Proceedings of the Pakistan Statistical Association	Pakistan
Proc. Qual. Contr. Ass., Bangalore	Proceedings of the Quality Control Association, Bangalore	India
Proc. R. Irish Acad.	Proceedings of the Royal Irish Academy	Ireland
Proc. Roy. Soc. (A & B)	Proceedings of the Royal Society (Series A & B)	Great Britain
Proc. Roy. Soc. N.S.W.	Proceedings of the Royal Society of N.S. Wales	Australia
Proc. Roy. Soc. N.Z.	Proceedings of the Royal Society of New Zealand	New Zealand
Proc. Soc. Exp. Biol., N.Y.	Proceedings of the Society of Experimental Biology and Medicine	U.S.A.
Proc. Soil Sci. Soc. Amer.	Proceedings, Soil Science Society of America	U.S.A.
Przegl. Statyst.	Przeglad Statystyczny	Poland
Psychometrika	Psychometrika	U.S.A.
Public Health Rep.	Public Health Reports	U.S.A.
Public Opinion Quart.	Public Opinion Quarterly	U.S.A.
Publ. Inst. Mat. Estadist. Fac. Ingen. Agric., Montevideo	Publicaciones del Instituto de Matemáticas y Estadística de la Facultad de Ingenieria y Agrimensura, Montevideo	Uruguay
Publ. Inst. Statist. Paris	Publications de l'Institut de Statistique de l'Université de Paris	France
Publ. Math., Debrecen	Publicationes Mathematicae	Hungary
Publ. Math. Inst. Hung. Acad. Sci.	Publications of the Mathematical Institute of the Hungarian Academy of Sciences	Hungary
Quart. J. Econ.	Quarterly Journal of Economics	U.S.A.
Quart. J. Exp. Physiol.	Quarterly Journal of Experimental Physiology	U.S.A.
Quart. J. Math.	Quarterly Journal of mathematics, Oxford	Great Britain
Quart. J. Med.	Quarterly Journal of Medicine	Great Britain
Quart. J. R. Met. Soc.	Quarterly Journal of the Royal Meteorological Society	Great Britain

RAND Corp., Publ. & Rep.	RAND Corporation, Publications and Reports	U.S.A.
R.C. Accad. Lincei	Rendiconti dell'Accademia Nazionale dei Lincei	Italy
R.C. Circ. Mat.	Rendiconti del Circolo Matematico	Italy
R.C. Mat. Univ. Roma	Rendiconti di Matematica e delle sue applicazioni	Italy
R.C. Semin. Sci. Cagliari	Rendiconti del Seminario della Facoltà di Scienze, Cagliari (University)	Italy
Rep. Statist. Appl. Res. (JUSE)	Reports of Statistical Application Research, Union of Japanese Scientists & Engineers	Japan
Research, Beirut	Research, A.U.B.—Beirut	Lebanon
Rev. Acad. Madrid	Revista de la Real Academia de Ciencias Exactas	Spain
Rev. Brasil. Estatist.	Revista Brasileira de Estatística	Brazil
Rev. Cienc. Tucumán	Revista de Ciencias de la Universidad Nacional del Tucumán	Argentine
Rev. Écon.	Revue Économique	France
Rev. Econ. Statist.	Review of Economics and Statistics	U.S.A.
Rev. Econ. Stud.	Review of Economic Studies	Great Britain
Rev. Economia, Lisbon	Revista de Economia	Portugal
Rev. Estocást. Soc. Argent. Estadíst.	Revista Estocástica de la Sociedad Argentina de Estadística	Argentine
Rev. Fac. Cienc. Econ. Admin. Montevideo	Revista de la Facultad de Ciencias Economicas y de Administracion, Montevideo	Uruguay
Rev. Int. Statist. Inst.	Review, International Statistical Institute	Netherlands
Rev. Mat. Hisp.-Amer.	Revista Matematica Hispano-Americana	Spain
Rev. Rech. Operat.	Revue de Recherche Operationelle	France
Rev. Sci., Paris	Revue Scientifique	France
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Rev. Statist. Appl.	Revue de Statistique Appliquée	France
Ric. Sci.	Ricerca Scientifica	Italy
Riv. Int. Sci. Soc.	Rivista Internazionale di Science Sociali	Italy
Riv. Ital. Econ. Demogr. Statist.	Rivista Italiana di Economia Demografia e Statistica	Italy
Riv. Polit. Econ.	Rivista di Politica Economia	Italy
Sankhyā	Sankhyā ; Indian Journal of Statistics	India
Schmollers Jb.	Schmollers Jahrbuch	Germany
Schr. Ver. Sozialpolit.	Schriften des Vereins für Sozialpolitik	Germany
Schweiz. Zeit. Volkwirtsch. Statist.	Schweizerische Zeitschrift für Volkwirtschaft und Statistik	Switzerland
Science	Science	U.S.A.
Science & Culture	Science and Culture	India
Scientia	Scientia	Italy
Scottish J. Polit. Econ.	Scottish Journal of Political Economy	Great Britain
Simon Stevin	Simon Stevin, Groningen	Netherlands
Skand. Aktuartidskr.	Skandinavisk Aktuarietidskrift	Sweden
Sociometry	Sociometry	U.S.A.
Soil Sci.	Soil Science	U.S.A.
South Africa J. Econ.	South African Journal of Economics	South Africa
Statistica, Bologna	Statistica	Italy
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Statist. Neerlandica	Statistica Neerlandica, The Hague	Netherlands
Statist. Praxis	Statistische Praxis	Germany
Statist. Qual. Contr.	Statistical Quality Control	Japan
Statist. Rev. Belgrade	Statisticka Revija	Yugoslavia
Studi Mercato	Studi di Mercato	Italy
Studia Math.	Studia Mathematica	Poland
Sugaku	Sugaku (Mathematical Society of Japan)	Japan
Tech. News, Nat. Bur. Stand.	Technical News Letter, National Bureau of Standards, Washington	U.S.A.
Tech. Publ., Amer. Soc. Test. Mat.	Technical Publications, American Society for Testing Materials	U.S.A.
Tech. Rep., C.S.I.R.O., Aust.	Technical Reports, Commonwealth Scientific & Industrial Research Organisation	Australia

Technometrics	Technometrics	U.S.A.
Teor. Veroyat. Primen.	Teoriia Veroyatnostey i ee Primenenia	U.S.S.R.
Textile Res.	Textile Research	U.S.A.
Tobacco Sci.	Tobacco Science	U.S.A.
Trab. Estadist.	Trabajos de Estadística	Spain
Trans. Edin. Math. Soc.	Transactions of the Edinburgh Mathematical Society	Great Britain
Trans. Fac. Actuar., Edinb.	Transactions of the Faculty of Actuaries	Great Britain
Trans. Manchester Statist. Soc.	Transactions of the Manchester Statistical Society	Great Britain
Trans. Nat. Inst. Sci. India	Transactions of the National Institute of Sciences of India	India
Trans. N.Y. Acad. Sci.	Transactions of the New York Academy of Science	U.S.A.
Trans. Roy. Soc. N.Z.	Transactions of the Royal Society, N. Zealand	New Zealand
Trans. Soc. Actuar., Chicago	Transactions of the Society of Actuaries	U.S.A.
Ukrain. Mat. Z.	Ukrainskij Matematiceskij Zurnal	U.S.S.R.
U.N. (F.A.O.) Tech. Publ.	United Nations (Food and Agricultural Organisation) Technical Publications	Italy
U.N. Statist. Publ.	United Nations : Statistical Office Publications	U.S.A.
Unternehmensforschung	Unternehmensforschung (Operations Research)	Germany
U.S. Census Publ.	U.S. Bureau of the Census, Publications	U.S.A.
U.S. Dept. Agric., Publ.	U.S. Department of Agriculture, Publications	U.S.A.
Verh. Kon. Belg. Akad., II	Verhandelingen. Koninklijke Belgische Akademie (Serie 2)	Belgium
Verh. Vlaamse Akad. Belg.	Verhandelingen. Koninklijke Vlaamsche Akademie	Belgium
Vitamins & Hormones	Vitamins and Hormones	U.S.A.
Weltwirts. Arch.	Weltwirtschaftliches Archiv	Germany
Wirtsch. und Statist.	Wirtschaft und Statistik	Germany
Yorkshire Bull. Econ. Soc. Res.	Yorkshire Bulletin of Economic and Social Research	Great Britain
Zastosowania Mat.	Zastosowania Matematyki	Poland
Zeit. angew. Math. Mech.	Zeitschrift für angewandte Mathematik und Mechanik	Germany
Zeit. ges. Staatswiss.	Zeitschrift für die gesamte Staatswissenschaft	Germany
Zeit. Handelswiss. Forsch.	Zeitschrift für Handelswissenschaftliche Forschung	Germany
Zeit. Nat.-Ökon.	Zeitschrift für Nationalökonomie	Austria

The short titles given above are based upon the principles stated in the League of Nations report *International Code of Abbreviations for Titles of Periodicals* (1930) and its *Supplement* (1932) as also used in the *World List of Scientific Periodicals*

BIOMETRIKA

VOL. 47, PARTS 1 AND 2 JUNE 1960

CONTENTS

Memoirs

BARTLETT, M. S., GOWER, J. C., and LESLIE, P. H. A comparison of theoretical and empirical results for some stochastic population models

KENDALL, DAVID G. Birth-and-death processes, and the theory of carcinogenesis

ASHFORD, J. R., SMITH, C. S., and BROWN, SUSANNAH. The quantal response analysis of a series of biological assays on the same subject

WALKER, A. M. Some consequences of superimposed error in time series analysis

FINCH, P. D. Deterministic customer impatience in the queueing system GI/M/1

BARTON, D. E., DAVID, F. N., and FIX, EVELYN. The polykays of the natural numbers

SRIVASTAVA, A. B. L. The distribution of regression coefficients in samples from bivariate non-normal populations. I. Theoretical investigation

KASTENBAUM, MARVIN A. The separation of molecular compounds by countercurrent dialysis: a stochastic process

SAW, J. G. A note on the error after a number of terms of the David-Johnson series for the expected values of normal order statistics

WATSON, G. S. More significance tests on the sphere

JOHNSON, N. L. An approximation to the multinomial distribution: some properties and applications

VAGHOLKAR, M. K., and WETHERILL, G. B. The most economical binomial sequential probability ratio test

McGREGOR, J. R. An approximate test for serial correlation in polynomial regression

BERKSON, JOSEPH. Nomograms for fitting the logistic function by maximum likelihood

HAIGHT, FRANK A., and BREUER, MELVIN ALLEN. The Borel-Tanner distribution

BURR, E. J. The distribution of Kendall's score S for a pair of tied rankings

Miscellanea

Contributions by D. H. BHATE, C. R. BLYTH and C. L. CURME, C. BURROWS, C. W. CLUNIES-ROSS and R. H. RIFFENBURGH, P. D. FINCH, R. G. LAHA and E. LUKACS, P. H. LESLIE, H. D. PATTERSON, B. R. RAO, G. L. TURIN, C. A. WILKINS, M. E. WISE

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PRINCIPAL CONTENTS OF RECENT ISSUES

Vol. 121—Part 4, 1959

The economics of housing and urban development. By P. A. STONE. (With Discussion)

Simulation studies of industrial operations. A Discussion opened by P. V. YOULE, K. D. TOCHER, W. N. JESSOP and F. I. MUSK

The 125th Anniversary of the Royal Statistical Society
Annual Report of the Council

Vol. 122—Part 1, 1960

Statistics as a tool of management. The Inaugural Address of the President, SIR HUGH BEAVER, K.B.E. (With Proceedings)

Practical experimentation as a teaching method in statistics. By G. H. JOWETT and HILDA M. DAVIES. (With Discussion)

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BIOMETRICS

Journal of the Biometric Society

VOL. 16, NO. 1 MARCH 1960

CONTENTS

A Method of Studying Manner of Growth	*S. C. Pearce*
Analysis of Covariance for a 3×4 Triple Rectangular Lattice Design (3 associate p.b.i.b.)	*Bernard S. Pasternack*
Competition in Populations Consisting of One Age Group	*F. Sogaard Andersen*
A Synthesis of Multivariate Techniques to Distinguish Patterns of Growth in Grasshoppers	*R. E. Blackith*
A Significance Test for the Separation of Two Highly Multivariate Small Samples	*A. P. Dempster*
An Ecological Distribution Akin to Fisher's Logarithmic Distribution	*J. H. Darwin*
On the Number of Self-Incompatibility Alleles Maintained in Equilibrium by the Gi en Mutation Rate in a Population of Given Size: A Re-examination	*Sewall Wright*
Ties in Paired-Comparison Experiments using a Modified Thurstone-Mosteller Model	*W. A. Glenn and H. A. David*
A Statistical Model for Diagnosing Zygosis by Ridge-Count	*Donald L. Richter and Seymour Geisser*

QUERIES AND NOTES

On a 5×2^2 Factorial Design	*B. V. Shah*
On Combining Partial Correlation Coefficients	*D. J. Finney*
On a Test for Order	*J. B. Chassan*
Partitioning the Between Slopes Sum of Squares for Forest Growth Data	*R. M. Cormack*
Tables for W. L. Steven's Asymptotic Regression	*H. Linhart*

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JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION

VOLUME 55 JUNE 1960 NUMBER 290

BECHHOFER, ROBERT: A MULTIPLICATIVE MODEL FOR ANALYZING VARIANCES WHICH ARE AFFECTED BY SEVERAL FACTORS

COHEN, A. CLIFFORD, JR.: ESTIMATION IN THE TRUNCATED POISSON DISTRIBUTION WHEN ZEROS AND SOME ONES ARE MISSING

HAYS, WILLIAM L.: A NOTE ON AVERAGE TAU AS A MEASURE OF CONCORDANCE

HOGG, ROBERT V.: CERTAIN UNCORRELATED STATISTICS

MAISAL, SHERMAN J.: CHANGES IN THE RATE AND COMPONENTS OF HOUSEHOLD FORMATION

LORD, FREDRIC M.: LARGE-SAMPLE COVARIANCE ANALYSIS WHEN THE CONTROL VARIABLE IS FALLIBLE

MUTH, JOHN F.: OPTIMAL PROPERTIES OF EXPONENTIALLY WEIGHTED FORECASTS

QUANDT, RICHARD E.: TESTS OF THE HYPOTHESIS THAT A LINEAR REGRESSION SYSTEM OBEYS TWO SEPARATE REGIMES

RIDER, PAUL R.: VARIANCE OF THE MEDIAN OF SAMPLES FROM A CAUCHY DISTRIBUTION

SPIEGELGLAS, STEPHEN: A STATISTICAL INVESTIGATION OF THE INDUSTRIALIZATION CONTROVERSY

BOOK REVIEWS

AMERICAN STATISTICAL ASSOCIATION
1757 K STREET, N. W.
WASHINGTON 6, D. C.

AUTHOR INDEX

VOL. 1: No. 3 (327-532)

Adam, A. 2.1
Aggarwal, O. P. 2.8
Agnew, R. P. 1.5
Aitchison, T. 2.8
Akaike, H. 1.5, 10.2
Amato, V. 0.6, 2.5
Anderson, R. L. 9.6
Aoyama, H. 6.4
Armitage, P. 6.8
Aroian, L. A. 8.9
Ashford, J. R. 6.8
Azorín, F. 4.3, 6.0

Barakrishnan, A. V. 10.7
Barton, D. E. 1.3, 1.3
Bellman, R. 11.5
Bharucha-Reid, A. T. 1.0, 1.5
Bodmer, W. F. 2.9, 5.5
Bose, R. C. 9.3, 9.3
Bradley, R. A. 9.9
Brandwood, L. 2.8
Braumann, P. 1.1
Burkholder, D. L. 2.0

Cansado, E. 10.0
Carter, R. L. 9.3
Certaine, J. 2.6
Chanda, K. C. 5.0
Chernoff, H. 9.7
Cherubino, S. 0.6
Clemans, K. G. 4.4
Ciucu, G. 1.5
Cox, D. R. 2.8
Crow, E. L. 4.4

Dall'aglio, G. 6.0
Danzig, D. van 1.8
David, Florence N. 1.3, 1.3, 6.1, 7.5
David, H. T. 8.8
Deal, R. B. 4.1
Draper, N. R. 9.3
Dreyfus, S. 11.5

Eeden, Constance van 3.2

Fay, E. A. 8.8
Firescu, D. 10.2
Fisz, M. 1.5
Frazier, D. 8.9
Fréchét, M. 1.6
Fuchs, A. 2.1, 10.1

Gabriel, K. R. 10.1
Gardner, R. S. 4.4
Geisser, S. 5.8
Ghosal, A. 10.4
Gilbert, E. J. 10.2
Gilbert, E. N. 1.3
Giri, N. C. 9.1
Graybill, F. A. 9.1
Green, B. F. Jr. 11.5
Gridgeman, N. T. 9.5
Guttman, Ruth 6.7
Guttman, L. 2.8, 4.5, 6.5, 6.7

Haga, T. 6.1
Haight, F. A. 10.4
Hall, W. J. 7.7
Harris, B. 1.2
Harris, E. K. 6.8
Harter, H. L. 3.8
Hartley, H. O. 9.3
Hatori, H. 10.5
Hayashi, C. 10.9
Healey, M. J. 11.5
Henderson, C. R. 7.3
Heumann, H. 2.3
Hooper, J. W. 4.1

Ihm, P. 1.0
Isii, K. 1.5, 5.6, 10.5

Jílek, M. 4.5
Johnson, N. L. 2.5
Joshi, D. D. 10.5

Kac, M. 10.1
Kallianpur, G. 10.2
Kao, J. H. K. 2.5
Kapur, M. N. 5.4
Kempthorne, O. 7.3
Kimball, A. W. 2.2
Klem, L. 11.5
Konijn, H. S. 5.2, 5.6
Kramer, H. P. 10.1
Krelle, W. 0.8
Krosigk, C. M. von 7.3
Kruskal, W. H. 6.9
Kuiper, N. H. 3.0, 5.6
Kulldorf, G. 4.3

Laha, R. G. 1.2, 2.1
Lancaster, H. O. 6.2
Laubscher, N. T. 2.2
Leach, E. 2.2
Levert, C. 4.5
Lieblein, J. 9.2
Líkar, O. 4.5
Lomnicki, Z. A. 10.6
Lukacs, E. 1.2
Luke, Y. L. 0.4
Mallows, C. L. 4.1
Mickey, M. R. 8.2
Mihoc, G. 4.3
Milne, A. 8.6
Mitra, S. K. 7.4
Mode, C. J. 7.2
Moran, P. A. P. 6.0, 10.9
Morley-Jones, R. 9.1
Morrison, D. 0.3
Mortara, G. 11.8
Moshman, J. 11.7
Motoo, M. 11.7
Muller, M. E. 2.3

Nasr, S. K. 10.4
Nass, C. A. G. 6.9
Neustadt, L. 0.6
Nisselson, H. 8.5
Nobis, E. 2.3
Noether, G. E. 7.5
Nordskog, A. W. 9.4

Oderfeld, J. 8.8
Onicescu, O. 1.0, 10.0
Onoyama, T. 10.6
Ostle, B. 6.2

Pachares, J. 3.3
Panizzon, G. 6.1, 11.0
Pawlik, E. 6.9
Pearson, E. S. 3.2
Pillai, K. C. S. 3.3, 3.3
Praporgescu, N. 10.1, 10.1
Pratt, J. W. 1.6

Quenouille, M. H. 9.2

Ralston, A. 0.3
Rendel, J. M. 9.2
Rényi, A. 10.1
Révész, P. 10.0
Roberts, S. W. 8.9
Robertson, A. 7.2
Robinson, H. F. 7.2
Rubin, H. 10.2
Rybarz, J. 3.1

Sakino, S. 10.9
Salton, G. 0.3
Sankaranarayanan, G. 1.5
Savage, I. R. 8.8
Scarf, H. 0.3
Schmetterer, L. 10.5
Schumann, D. E. W. 9.9

Schwarz, K.	11.8	Taga, Y.	10.5	Walsh, J. E.	5.5, 8.8
Searle, S. R.	7.3	Tedeschi, B.	11.8	Ward, D. H.	11.3
Sheldon, J. W.	0.6	Teghem, J.	6.4	Wartmann, R.	7.0
Shrikande, S.	9.1	Terry, M. E.	0.2	Watson, G. S.	3.1
Sibuya, M.	6.1	Theil, H.	4.1	Wearden, S.	9.4
Siotani, M.	5.5	Theodorescu, R.	1.1	Weeks, D. L.	9.1
Slepian, D.	10.1	Tienzo, B. P.	3.3	Weichselberger, K.	6.9, 6.9
Smith, J. E. K.	11.5	Trybula, S.	0.7, 1.0, 4.0, 8.8	Weinstein, A. S.	10.7
Smith, R. B.	0.6	Tucker, H. G.	10.2	Weir, J. B. de V.	3.1
Srivastava, A. B. L.	7.5			Wette, R.	6.0
Steck, G. P.	6.2			Whitman, D. R.	8.9
Steel, R. G. D.	7.8	Uematu, T.	6.4	Wilkinson, J. H.	0.6
Stein, C.	4.4	Uhlmann, W.	5.6	Wilkinson, R. A.	4.2
Steinhaus, T. D.	0.7	Urbanik, K.	10.0	Wilks, S. S.	2.6
Sterling, T. D.	5.0			Williams, E. J.	5.8
Steutel, F. W.	10.4			Winsten, C. B.	10.4, 10.4
Steyn, H. S.	5.2	Vagholkar, M. K.	8.8	Woolsey, T. D.	8.5
Stoller, D. S.	10.2	Van Elteren, P.	7.7		
Stoller, L. C.	10.2	Vas, É.	5.7	Zaremba, S. K.	10.6
Streibel, Charlotte T.	10.2	Veen, B.	8.1	Zia-ud-din, M.	3.0
Suzuki, Y.	0.8	Vince, I.	5.6	Zubrcycki, S.	10.5

Correction Slip Abstract number 342

THE ANNALS OF MATHEMATICAL STATISTICS

VOL. 30, NO. 4—DECEMBER 1959

CONTENTS INCLUDE

Some Validity Criteria for Statistical Inferences — ROBERT J. BUEHLER
Conditional Confidence Level Properties — DAVID L. WALLACE
An Example of Wide Discrepancy between Fiducial and Confidence Intervals — CHARLES STEIN
Optimum Invariant Tests — E. L. LEHMANN
The Weighted Compounding of Two Independent Significance Tests — M. ZELEN and L. S. JOEL
Bayes Acceptance Sampling Procedures for Large Lots — D. GUTHRIE, JR. and M. V. JOHNS, JR.
Optimum Tolerance Regions and Power when Sampling from some Non-normal Universes — IRWIN GUTTMAN
Properties of Model II-Type Analysis of Variance Tests, A: Optimum Nature of the F-Test for Model II in the Balanced Case — LEON H. HERBACH
Some Remarks on Herbach's Paper, " Optimum Nature of the F-Test for Model II in the Balanced Case " — WERNER GAUTSCHI
The Most-Economical Character of some Bechhofer and Sobel Decision Rules — WM. JACKSON HALL
The Admissibility of Pitman's Estimator of a Single Location Parameter — CHARLES STEIN
The Use of Sample Quasi-Ranges in Estimating Population Standard Deviation — H. LEON HARTER
The Joint Cumulants of True Values and Errors of Measurement — FREDERIC M. LORD
Some Tests of Permutation Symmetry — R. WORMLEIGHTON
Contributions to the Theory of Rank Order Statistics—The One-Sample Case — I. RICHARD SAVAGE
The Distribution of a Generalized D_n^+ Statistic — MEYER DWASS
Null Distribution of the Hodges Bivariate Sign Test — JEROME KLOTZ
Exact Nonparametric Tests for Randomized Blocks — JOHN E. WALSH
A Generalization of Partially Balanced Incomplete Block Designs — B. V. SHAH
The Non-Existence of certain PRIB Designs — MANOHAR NARHAR VARTAK
A Necessary Condition for Existence of Regular and Symmetrical Experimental Designs of Triangular Type, with Partially Balanced Incomplete Blocks — JUNJIRO OGAWA
Optimal Spacing in Regression Analysis — H. A. DAVID and BEVERLY E. ARENS
Third Order Rotatable Designs for Exploring Response Surfaces — D. A. GARDINER, A. H. E. GRANDAGE and R. J. HADER

AMATO, V. (University of Catania)
A practical method for the inversion of Leontief's matrix—*In Italian* *Industria* (1959) 297-313 (11 refs.)

0.6 (–.–)

This article demonstrates a method proposed by the author for obtaining the inverse of Leontief's matrix.

After having stated the conditions for a matrix to be in conformity to Leontief's definition, the author demonstrates the many difficulties which increase with the order of the matrix. He suggests decomposing the matrix into factors easily invertible and so arranged as to give the desired solution.

The difficulty of the calculations that are required by the method may be met by the use of suitable electronic computers.

(P. Bandettini)

327

CHERUBINO, S. (University of Pisa)
On Leontief's matrices and on a problem in linear programming—*In Italian* *Industria* (1959) 156-164 (11 refs.)

0.6 (0.8)

The author, starting from the properties of matrices applied to the input-output analysis, introduces some variants in the demonstration and in the notation. He also considers a problem in linear programming.

A solution is outlined which gives the maximum and minimum of two (dual) homogeneous linear functions. Of these two functions, one is the total production of the n sectors and the other cost prices. The production and price functions describe two convex polyhedral cones, the equations of which have the same Leontief's matrix as the coefficients of the inequalities conditioning the maximum and the minimum.

This problem suggests a general method for the inversion of Leontief's matrices.

(P. Bandettini)

328

KRELLE, W. (Universität Bonn) **0.8** (–.–)

Discrete programming: theory and practical applications—*In German*
Unternehmensforschung (1958) **4**, 161-175 (10 refs.)

Discrete programming is defined as the solution of extremal problems with equations or inequalities as constraints under the additional condition of discrete values for some or all variables. Many economic problems take this form. The statement of the problem and its mathematical formulation are given for the following examples:

(i) planning of production with indivisible final products,
(ii) planning of production with indivisible factors,
(iii) planning of production with indivisible factors and final products,
(iv) planning of production with growing increase of profits,
(v) assignment problem,
(vi) localising problem,
(vii) travelling salesmen problem,
(viii) transportation problems: on the one hand for non-homogeneous goods and constant capacity of transportation units; on the other hand for different units with limited capacity and given frequency of transportation of persons,
(ix) allocation of machines.

Some methods of solving discrete programming problems are roughly outlined. The logical structure is always the same: from a certain number of combinations, that number will be chosen which yields an optimum with respect to a given criterion. Because of the immeasurably great number of possible combinations, one generally has at first to try to reduce as much as possible the number of variables and combinations by making use of the peculiarities of the special problem.

It is often advantageous for discrete programming to use only such variables which assume values of 0 or 1. If the original mathematical formation does not have this form, it can be obtained by substitution: this way was chosen by Markowitz and Manne. In this way the possible combinations will be reduced, on the one hand by a definition of the maximum and minimum number of those variables which must be in the solution and, on the other hand, by determining those variables which are certainly not included in the solution, as well as those which definitely are included.

If the constraints are not very important, the " approximation from above " is applied. First a discrete solution is determined without taking account of the constraints. After that one has to examine whether the constraints are fulfilled, and to resort to the second best value of the extremal function if this is not so, until a solution is obtained for which the constraints are satisfied.

If the constraints are essential, a large number of combinations become irrelevant. In this case there exist two ways of " approximation from below " to find all combinations which satisfy all constraints. The value of the extremal function has to be computed step by step after which the solution may be selected.

329 (H. Kregeloh)

LUKE, Y. L. (Midwest Research Institute, Kansas City) **0.4** (11.6)

Expansion of the confluent hypergeometric function in series of Bessel functions—*In English*
Math. Tab., Wash. (1959) **13**, 261-271 (13 refs.)

The confluent hypergeometric function, defined by

$$\phi(a, c, z) \equiv {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(c)_k k!},$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$, and where $\Gamma(z)$ is the gamma function, is expanded in a series of Bessel functions of integral order whose argument is independent of a and c. This expansion is useful for computational purposes since the parameters and variable of the confluent hypergeometric function appear separately. Bessel functions of integral order are easy to generate on a computer and tables are available for desk calculation.

If c is equal to $a+1$ the function reduces to the incomplete gamma function. The author develops several expansions for the incomplete gamma function and by a further specialisation of $a = \frac{1}{2}$, obtains expansions for the normal integral.

The author gives several numerical examples to illustrate the computational efficiency of the expansions.

(D. Teichroew)

MORRISON, D. (Space Tech. Labs., Los Angeles) **0.3** (–.–)

Numerical quadrature in many dimensions—*In English*

J. Ass. Comp. Mach. (1959) **6**, 219-222 (2 refs.)

Consider the problem of evaluating I, where I is given by

$$I = \int_{-a}^{a} \cdots \int_{-a}^{a} f(x_1, \ldots, x_n) \prod_{i=1}^{n} \omega(x_i) \prod_{i=1}^{n} dx_i$$

and where $\omega(x)$ is a positive and even weighting function. A numerical integration formula is developed, the essential step being the reduction of the problem to one of approximating integrals of functions with fewer variables. A k-dimensional trace of a function of n variables is defined as the value of the function obtained by setting all variables equal to zero except k of them, e.g., the two-dimensional trace $P(x_1, x_3)f$ is $f(x_1, 0, x_3, 0, \ldots, 0)$. It is then proved that if f is a sum of functions of d variables then f may be written in terms of traces of these variables. The numerical quadrature formula is developed explicitly for the case of two dimensions and a fifth order formula is obtained. By writing the function as a sum of its traces, I is reduced, in this case, to a problem of numerical quadrature on a square or on a line segment. Both of these problems are solved with the aid of a three-point Gauss quadrature formula which corresponds to the weighting function $\omega(x)$, there the three points are $(-\alpha, 0, \alpha)$ with weights $(\omega_1, \omega_0, \omega_1)$.

The formula becomes

$$I = A^{n-2}\omega_1^2 \sum_{j=1}^{n-1} \sum_{j=i+1}^{n} \sum_{p,q=1}^{2} P(x_i^{(p)}, x_j^{(q)}) f$$

$$+A^{n-2}\omega_1[-2\omega_1 n+\omega_0+4\omega_1] \sum_{i-1}^{n} \sum_{p-1}^{2} P(x_i^{(p)})f$$

$$+A^{n-2}[2\omega_1^2 n^2-2\omega_1(\omega_0+3\omega_1)n+A^2]Pf,$$

where $x_i^{(1)} = -\alpha$ and $x_i^{(2)} = \alpha$ and $Pf = f(0, 0, \ldots, 0)$.

In the special case where $\omega(x)$ is equal to 1, $A = 2$, we have $\alpha = 3/5$, $\omega_0 = 8/9$, $\omega_1 = 5/9$; e.g., the points and weights of the Gauss-Legendre quadrature formula. In this case the result reduces to one already given by Hammer and Stroud [Technical Report No. 5, Numerical Analysis Laboratory, University of Wisconsin].

(D. Teichroew)

NEUSTADT, L. (Reeves Instrument Corp., N.Y.) **0.6** (11.6)

A method for computing eigenvectors and eigenvalues on an analogue computer—*In English*

Math. Tab., Wash. (1959) **13**, 194-201 (8 refs., 2 tables, 1 fig.)

The problem of computing eigenvectors and eigenvalues of an arbitrary matrix has been the subject of a number of papers in recent years. Most of the methods suggested have dealt with the use of digital computers; this paper gives a procedure using analogue equipment.

The technique is based on the analogue computer method for solving problems in linear programming described by Pyne [*Trans. Amer. Instn. Elect. Engrs.*, Part 1—*Communications and Electronics* (1956), 139-143]. The method consists of formulating the problem of determining eigenvalues as an extremum problem, as follows: let x be the radius vector to the point $X = (x_1, \ldots, x_n)$ in n space. Let $x_1, x_2, \ldots, x_n$ be functions of time. The first eigenvector $\mathbf{e}^1$ is found by letting the point X move on the unit sphere until it reaches a steady state corresponding to a maximum of the function $AX.X$. Then $\mathbf{e}^1$ equals the steady state of X. The other eigenvectors, $\mathbf{e}^k$, are found in the same way using additional restrictions on X such that $X.\mathbf{e}^1 = X.\mathbf{e}^2 = \ldots = X.\mathbf{e}^{k-1} = 0$.

The analogue method described has the advantage of yielding both eigenvectors and eigenvalues of a real symmetric or complex Hermitian matrix, and do not require a trial and error procedure. This is accomplished at the expense of additional computing equipment. The paper gives the amount of computing equipment required to compute an eigenvector. The amount of equipment depends on the size of the matrix and the number of eigenvectors desired. The eigenvalues can be obtained using the computer or can be obtained by hand. Once an n by n matrix problem has been programmed, this programming and the associated patching can be used for any other n by n matrix with only minor changes.

The author gives two examples, one a 3 by 3 matrix and the other a 6 by 6 matrix, and shows the difference between the analogue and the correct results. Three place accuracy can be obtained for the eigenvector and if the eigenvalue is computed by hand six place accuracy can be obtained.

(D. Teichroew)

RALSTON, A. (Bell Telephone Labs., Whippany, N.J.) **0.3** (–.–)

A family of quadrature formulas which achieve high accuracy in composite rules—*In English*

J. Ass. Comp. Mach. (1959) **6**, 384-394 (2 refs.)

A class of quadrature formulas is derived which achieve higher accuracy in composite rules (i.e., where the interval of integration is broken up into a number of sub-intervals) than analogous Newton-Cotes or Gaussian formulas. The cost of this higher accuracy is the computation of one or two more ordinates over the whole interval of integration.

The high accuracy is obtained by using Gaussian techniques in the interior of each sub-interval and by using the end-points of each sub-interval as abscissas with weights of equal magnitude and opposite sign. In this way when the sub-intervals are put together only the end-points of the whole interval of integration remain. It is proved that the abscissas are all real and interior to the sub-interval and that the weights corresponding to the interior abscissas are positive.

Since the abscissas are not equally spaced, the method is not suited to tabular functions but rather to analytically given functions. The round-off properties of the formulas are discussed and are shown to be quite good.

(A. Ralston)

SALTON, G. (Computation Lab., Harvard Univ., Boston) **0.3** (2.0)

The use of the central limit theorem for interpolating in tables of probability distribution functions—*In English*

Math. Tab., Wash. (1959) **13**, 213-216 (6 refs.)

It is well known that interpolation in tables of probability density and distribution functions is difficult, especially if the values desired are in the "tails". It is possible to make use of the central limit theorem to increase accuracy of interpolation for those distributions which tend to normality.

As an example, consider the problem of determining the value of the binomial distribution $E(n_0, r_0, p_0+\Delta p)$ where n_0 is the sample size, r_0 is the number of observed successes and $p_0+\Delta p$ is the parameter. Let us assume that the function is not tabulated for the argument $p_0+\Delta p$. The standard interpolating procedure consists of taking the tabulated values corresponding to arguments near the desired ones and constructing a polynomial based on $k+2$ given points. The desired value is then obtained by evaluating the polynomial at the desired argument. The method proposed in this paper is the following:

(i) Write down the functional values corresponding to the arguments given in the table.
(ii) Look up in a normal table the normal deviates corresponding to these probabilities.
(iii) Use the standard interpolating technique to interpolate in these normal deviates for the value of the normal deviate corresponding to the desired parameters; e.g., the argument of the interpolating polynomial would be $p+\Delta p$ in the above example.
(iv) Look up in the table of the normal distribution the value corresponding to this calculated normal deviate. This is the desired value.

A second approach suggested in this paper is to use, instead of the normal distribution in the above method, the logit function

$$L(x) = 1/1+e^{-\beta x}.$$

This function is relatively close to the normal distribution for particular values of β.

These two methods have been compared with the ordinary method using both two and four point interpolation, in which interpolation is performed for n_0 and r_0 as well as for p_0. The four point normal deviate procedure was far superior to other methods.

(D. Teichroew)

SCARF, H. (Stanford University, California) **0.8** (1.8)

Bayes solutions of the statistical inventory problem—*In English*

Ann. Math. Statist. (1959) **30**, 490-508 (8 refs.)

In the inventory problem a sequence of purchases are made at the beginning of a number of regularly spaced time periods. In addition to the purchasing cost, which may be a general function of the amount purchased, it is customary to charge a holding cost at the end of each period if excess stock is on hand or a shortage cost if there is a deficit of stock. Demands are usually assumed to be independently distributed and with a common known distribution. The solution to the problem is a set of purchasing decisions which minimises total expected cost: sometimes the cost is discounted.

In this paper we remove the assumption that the demand is known, and replace it by the assumption that the demand distribution contains an unknown parameter (specifically from the exponential family) and that the parameter has a known *a priori* Bayes distribution.

This feature adds considerable complexity and other features are simplified in order to obtain a manageable answer. The primary simplification is that the purchase cost is assumed to be proportional to the amount purchased.

It is then shown that the optimal policy is defined by a sequence of functions x_n (mean observed demand), such that if at the beginning of the nth period the stock level is less than x_n, the difference is purchased and otherwise no purchases are made.

The functions x_n are shown to be increasing in the mean demand, and it is also shown that purchases will never be made in a period if demand during the previous period has been too small. Finally an asymptotic expansion of x_n, for large n, is given.

(H. Scarf)

SHELDON, J. W. (Computer Usage Co., Inc., New York) **0.6** (11.5)

On the spectral norms of several iterative processes—*In English*

J. Ass. Comp. Mach. (1959) **6**, 495-505 (7 refs.)

A linear iterative process of first degree for solving a set of simultaneous linear equations

$$\mathbf{Au} = \mathbf{d},$$

where $\mathbf{A}$ is a symmetric, positive definite matrix of order n, and $\mathbf{u}$ and $\mathbf{d}$ are column vectors, is defined by

$$\mathbf{u}^{(m)} = H_m\mathbf{u}^{(m-1)} + M_m\mathbf{d}$$

where $$H_n + M_m\mathbf{A} = I, \quad m = 1, 2, \ldots$$

let $$K_m = \prod_{n=1}^{m} H_n.$$

The spectral radius of a square matrix $\mathbf{A}$ is equal to the maximum of the magnitudes of the characteristic roots of $\mathbf{A}$. The spectral norm is the positive square root of the spectral radium of $\mathbf{A}'\mathbf{A}$. There exists a class of iterative methods in which K_m is a polynomial $P_m(H)$ of degree m in a given matrix $\mathbf{H}$ which is independent of m. Among these iterative methods are the Jacobi method, Richardson's method, the " best-strategy " methods of Stiefel, the Young-Frankel successive over-relaxation method, and the linearly-accelerated Gauss-Seidel method.

This paper gives, under certain assumptions, formulas for the connection between the spectral radius of the Jacobi matrix and the spectral norms of four iterative processes.

The first process is the Young-Frankel successive over-relaxation method in which

$$K_m = (L_{\sigma_1,w_b})^m, \text{ where } L_{\sigma,w} = (I - wE)^{-1}[(1-w)I + wE']$$

where w = the value which minimises the spectral norm of $L_{\sigma,w}$. It is shown that the Euclidean norm of the error may actually increase in the first few iterations of the successive over-relaxation method. For the second process considered

$$K_m = (L_{\sigma_1,w_b})^{m-1}L_{\sigma_1,1}$$

it is shown that if the matrix $\mathbf{A}$ has σ_1 ordering, a smaller spectral norm will be obtained if the successive over-relaxation method for performing a single iteration with $w = 1$ is carried out first. The third process is defined by:

$$K_m = T_m[\{2/\rho(B)^2\}L_{\sigma_1,1} - I]/T_m[\{2/\rho(B)^2\} - 1].$$

where $T_m(x)$ are the Tchebycheff polynomials of the first kind. It is shown that the Euclidean norm of error goes to zero as for $m/T_m[(2/\rho(B)^2) - 1]$. For the fourth method

$$K_m = L_{\sigma_1,1}T_{m-1}[\{2/(\rho\beta)^2\}L_{\sigma_1,1} - I]/T_{m-1}[\{2/\rho(\beta)^2\} - 1].$$

it is shown that the Tchebycheff accelerated Gauss-Seidel method can be improved by performing a single iteration first with $w = 1$ and without any " acceleration."

(D. Teichroew)

SMITH, R. B. (Westinghouse Elec. Corp., Pittsburgh) **0.6** (1.3)

Two theorems on inverses of finite segments of the generalised Hilbert matrix—*In English*

Math. Tab., Wash. (1959) **13**, 41-43 (5 refs.)

The generalised Hilbert matrix is defined to be

$$H_n = (h_{ij}),\ h_{ij} = \frac{1}{p+i+j-1} \qquad i, j = 1, 2, \ldots, n;$$

$$p \neq -1, -2, \ldots, -(2n-1).$$

The elements of the inverse of this matrix can be given as a closed expression, in particular the element in the ith row and jth column of S_n is

$$S_n^{ij} = \frac{(1)^{i+j}}{p+i+j-1}\left[\frac{\prod_{k=0}^{n-1}(p+i+k)(p+j+k)}{(i-1)!\,(n-i)!\,(j-1)!\,(n-j)!}\right]$$

The following formulas are developed for sums of rows and columns and sums of all elements in the inverse.

$$\sum_{=1}^{n} S_n^{ij} = \sum_{i=1}^{n} S_n^{ji} = (-)^{n+j} \prod_{k=0}^{n-1} (p+j+k)/(j-1)!\,(n-j)!$$

$$\sum_{i,j=1}^{n} S_n^{ij} = n(p+n).$$

These results are proved using the relationships involving the binomial coefficients and then defining a function of x as a series, differentiating the function and setting x equal to 1.

(D. Teichroew)

337

STEINHAUS, H. & TRYBULA, S. (Math. Inst., Polish Academy of Sciences, Wroclaw) **0.7** (1.7)

Measurement by successive comparison—*In Polish*

Zastosowania Mat. (1959) **4**, 204-212

In this paper the authors give the solution, by the methods of the theory of games, of a few practical problems in the sphere of approximate measurements of certain quantities. For example, comparing successively an unknown quantity $x \in \langle 0, 1\rangle$ with known quantities $t_1, t_2, \ldots, t_n$. On the basis of the results of those comparisons we give the approximate value of x' of x. The functions $t_1, t_2, \ldots, t_n$ and the estimate x' should be chosen so as to minimise $W(|x-x'|)$. This problem may be regarded as a game of man against nature, in which man chooses $t_1, t_2, \ldots, t_n$ and x', and nature chooses x. Man pays to nature $W(|x-x'|)$. We prove that if W is an increasing function of the argument $|x-x'|$, then the best strategy for man in the minimax sense is $t_1 = \frac{1}{2}$; t_2 is equal to the centre of the segment $\langle 0, \frac{1}{2})$ or $(\frac{1}{2}, 1\rangle$, according to whether the point x is in the first or in the second of those intervals, etc., and x' is equal to the centre of the segment obtained after n steps. If x is a random variable with a uniform distribution, then the same strategy minimises the expected value of $W(|x-x'|)$.

A few practical consequences of those and similar results are given.

(H. Steinhaus)

SUZUKI, YUKIO (Inst. Statist. Math., Tokyo) **0.8** (–.–)

Note on linear programming—*In English*

Ann. Inst. Stat. Math., Tokyo (1959) **10**, 89-105 (11 figs.)

In this paper, the author treats some computational methods for linear programming. He raises the question whether there is any method using the optimal solution on a basis of an initial linear programming problem for solving various new problems which are derived by varying the values of coefficients of the restriction of the initial linear programming problem. The method used here is essentially the same with the Dantzig's simplex method.

In section 2, the case where the value of the constraint vector is changed is treated. In this case, the optimal basis B^0 of the initial problem is not always the one corresponding to an optimal solution of the new problem and furthermore it may not be feasible for the new problem. When it is feasible, then an optimal solution of the new problem is only obtained by further simplex algorithm. However, in case it is not feasible for the new problem, the author constructs a modified linear programming problem which has B^* as a feasible basis and whose optimal solution coincides with the one of the new problem, where B^* is constructed by changing signs of some vectors in B^0. The author says that, when the change of the constraint vector is not so great, the modified problem may be solved with fewer stages.

In section 3 the author considers the case where the coefficient matrix A of the linear equations or inequalities of the restriction is changed. He only discusses the special case where a single column of A is changed. He splits the problem into the following two cases. Case (i): the vector changed is contained in B^0. Case (ii): it is not contained in B^0. For the former case, he constructs a modified problem which has B^0 as a feasible basis and whose optimal solution coincides with the one of the new problem. For the latter case also, such a modified problem is considered.

In section 4, the change in coefficients of the objective function is discussed. The author states that here we have only to continue the algorithm.

In section 5 the case where only one further restrictive linear equation is added to the initial restriction is discussed. In this case, a feasible basis of the new problem is easily constructed by using B^0 and here also the repetition of algorithm is required. In this section a lemma is given concerning a condition for existence of a feasible solution of the new problem.

In section 6 the case is treated where only one of linear equations of the restriction is eliminated. The method is analogous to that in the previous section. In this section, a lemma is given concerning a necessary condition for the existence of an optimal solution of the new problem.

(T. Uematu)

TERRY, M. E. (Bell Telephone Labs., Murray Hill, New Jersey) **0.2** (11.5)

Polynomial curve fitting on electronic computers—*In English*

Nat. Conv. Trans., Amer. Soc. Qual. Contr. (1959), 565-573 (3 refs.)

Polynomial curve fitting is simplified by the use of available tables of orthogonal polynomials when the n values of the independent variables are equally spaced. When the values of the independent variable are not equi-spaced and n is large, construction of orthogonal polynomials is expensive and difficult even with large computers.

Computing difficulties arise mainly from the condition of the square matrix $\mathbf{A}$ of the normal equations $\mathbf{A}b = g$. The problem discussed is that of finding a method of transforming the independent variables such that a reasonable inverse, $\mathbf{C}$, of $\mathbf{A}$ is obtained. It is necessary to raise the question of deciding when a matrix $\mathbf{C}$ is the inverse of a matrix $\mathbf{A}$. Mathematically, the answer is trivial, $\mathbf{CA} = \mathbf{I}$. The answer is not trivial when $\mathbf{C}$ has been computed and the product $\mathbf{CA} = \mathbf{D}$ has been calculated. The elements d_{ij} are found to fall within $1 \pm E_{ii}$ on the diagonal and $0 \pm E_{ij}$ off the diagonal, where E_{ij} is a small positive number. If $\mathbf{C}_0$ is an approximate inverse of $\mathbf{A}$, then $b_0 = \mathbf{C}_0 g$ and $\mathbf{A}b_0 = g_0$. The two vectors with elements $g_{0i} - g_i$ or $(g_{0i} - g_i)/g_i$, where $g_i \neq 0$, have been suggested as criteria for deciding when sufficient accuracy has been achieved. The criterion might be the specification of an upper bound for any element of $g_{0i} - g_i$ or $(g_{0i} - g_i)/g_i$. No method is known for evaluating $\mathbf{C}_0$ element by element.

A transformation is needed which will yield a matrix which can be inverted, which yields equal precision in the estimates of all the coefficients, and which is nearly an orthogonal transformation. The transformation defined in terms of the Tchebysheff polynomials meets these criteria. These polynomials are

$$T_j(x) = \cos j \arccos x \qquad -1 \leqq x \leqq 1$$

or

$$T_j^*(x) = \cos j \arccos (2x-1) \qquad 0 \leqq x \leqq 1$$

Other properties of these polynomials are

$$\text{Max } T_j(x) = \left| \text{ Min } T_j(x) \right| = 1$$

within the interval $(-1, 1)$; these boundaries are attained for all values of j. The Tchebysheff polynomials have a higher average error in approximating a polynomial than the Legendre polynomials but have a lower maximum error.

Since the power series expression for $T_j(x)$ has alternate zero coefficients when $-1 \leqq x \leqq 1$, it is better to apply the initial transformation to the independent variable x' as

$$x = (2x' - x'_{\max} - x'_{\min})/(x'_{\max} - x'_{\min}).$$

Remarkable gains in accuracy are afforded by the use of this Tchebysheff transformation.

(D. W. Gaylor)

WILKINSON, J. H. (National Physical Lab., Teddington) **0.6 (11.5)**

Stability of the reduction of a matrix to almost triangular and triangular forms by elementary similarity transformations—*In English*

J. Ass. Comp. Mach. (1959) **6**, 336-359 (5 refs., 6 tables)

In the recent paper Givens [*J. Soc. Industr. Appl. Math.* (1958) **6**, 26-50] has described a method of reducing a general matrix to almost triangular form by a succession of simple unitary transformations. In the same paper he described a second sequence of unitary transformations for reducing the almost triangular matrix to a triangular matrix. This second transformation is performed in n stages and at each stage one eigenvalue and one eigenvector of the current transformed matrix (and hence of the original matrix) are determined, the vector being obtained as a by-product of the transformation itself.

In this paper the author describes two alternative methods for reducing a general matrix to almost triangular form by means of similarity transformations which are derived as the product of a number of simple elementary similarity transformations. Such transformations have been described elsewhere, but here the emphasis is on numerical stability and details are given of a method of organising the computation in fixed-point arithmetic so that rounding errors are kept to a minimum.

A similar alternative is described for the further reduction to triangular form and it is shown that this second transformation suffers from numerical instability which will sometimes render it valueless for finding the eigenvector. The same criticism applies with equal force to the process described by Givens for the reduction from almost triangular to triangular form. It is shown that the instability is of a similar nature to that analysed in a recent paper by Wilkinson on the calculation of the eigenvectors of triple diagonal forms. The method which is suggested for overcoming this instability has been programmed for DEUCE and has given good results in practice.

In general the methods of this paper involve considerably less computation and programming than those described by Givens; so much so, that on DEUCE they can be performed using double precision in about the same time as can the Givens transformations using single precision. Experience with the two methods has suggested that they give comparable accuracy when both are performed using single precision arithmetic. In the final section a numerical example is given showing results which are typical of those which have been obtained in general.

(J. H. Wilkinson)

Volume 1. Nos. 1 and 2. CORRECTIONS

To assist subscribers who may file their abstracts so as to lose the sequence of serial numbers, the author and main classification number will now be given for each correction in addition to the serial number. On this occasion these additional particulars are also given for the first list published in Volume 1, No. 2 (serial 164).

6	0.4	Kabe	
11	1.0	Bayes	
12	1.8	Braumann	
13	1.5	Erdos and ors.	
40	3.2	Srivastava	
48	4.3	Finney	
55	4.3	Patterson	
59	4.3	Tanner	
68	5.3	Gnedenko	
69	5.6	Ishii	
73	5.6	Salib	
119	10.5	Adam	
122	10.9	Bennett	
136	10.1	Lévy	
151	11.9	Barnard	
106	8.8	Stange	L.H.S. of first displayed formula should read : $\sigma^2(1+\frac{1}{2}k^2)/n$. Third line from end, fraction should read : n_M/n_L.
133	10.1	Koźniewska	Original paper in English.
148	10.1	Urbanik	Original paper in English.
184	1.5	Trotter	Abstractor's initial should be " K ".
209	4.7	DeGroot	Abstractor's initials should be " D. H."
210	4.4	Dunn	Square bracket added to left-hand edge of second component of inequality.
228	5.4	Gnanadesikan	Inequality in line 10 should be reversed.
232	5.0	Lipták	Col. 2, line 10: displayed formula should read : $\tilde{\mathscr{L}} = \phi\{\ldots\ldots\ldots\ldots\ldots\ldots\}$.
272	8.5	Sobol	Second figure of main classification should be bold face type.
283/4	9.4	Kiefer and or.	Author's address should read " Ithaca ".
308	10.9	Moran and or.	Delete : duplicate of No. 137.

AGNEW, R. P. (Cornell University, Ithaca, N.Y.) **1.5** (2.4)

Asymptotic expansions in global central limit theorems—*In English*

Ann. Math. Statist. (1959) **30**, 721-737 (7 refs.)

This paper is concerned with constants $C_n(p)$ defined by

$$C_n(p) = \int_{-\infty}^{\infty} | F_n(x) - \phi(x) |^p dx$$

where $\phi(x)$ is the Gaussian distribution function and $F_n(x)$ is the distribution function of the sum $(\xi_1+\xi_2+\ldots+\xi_n)/n^{\frac{1}{2}}$, each ξ_i being a random variable having a distribution function $F(x)$. The case $p = 2$ is of primary interest. The author cites references wherein it is shown that the $C_n(p)$ exist if $p>\frac{1}{2}$ and have limit zero in n. For $F(x)$ as the symmetric binomial distribution function and $F(x)$ as a uniform distribution function, first order asymptotic expressions for the $C_n(2)$ are given.

The primary purpose of this paper is to establish asymptotic expansions for the $C_n(2)$ of the form

$$C_n(2) = \sum_{i=1}^{k} D_i n^{-i} + 0(n^{-k-1}).$$

The author carefully investigates the conditions under which such expansions are valid. He has given a rather detailed description of the work which was carried out to obtain these expansions. This work also resulted in formulae for the D_i in terms of the moments of $F(x)$; these are shown. A portion of the calculations for D_i is separable into two cases: different approaches are needed depending on whether or not the magnitude of the characteristic function of $F(x)$ is or is not periodic.

As a special example the author considers the binomial distribution function. For the general, non-symmetric case only D_1 is computed, with an indication that further straightforward calculations of higher D_i are possible but tedious. The symmetric case is considered in rather greater detail, the first ten terms of the sequence $\{C_n(2)\}$ being explicitly given. In addition, an approximate formula for $C_n(2)$, apparently good for $n \geqq 5$ is obtained. The procedure for calculating the $C_n(2)$ exactly is derived and presented in a form which can be used for numerical work. The exact method is, however, much more laborious than the approximate one.

(D. H. Shaffer)

Abstractor's Note: the first term on the right of expression (7.61) should read $0{\cdot}09403\ 15973\ n^{-1}$.

(D. H. Shaffer)

AKAIKE, H. (Institute of Statistical Mathematics, Tokyo) **1.5** (2.2)

On a successive transformation of probability distribution and its application to the analysis of the optimum gradient method—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11**, 1-16 (8 refs., 4 figs.)

In this paper is defined a kind of transformation of probability distributions over a set of real finite numbers, say n. The transformation is such that, when we apply it to a distribution, the probability of each point is multiplied by a factor proportional to the square of the distance of the point to the mean of the distribution to give a new probability distribution. It is shown that by such transformation the variance of the distribution never decreases. From this fact it can be seen that if we consider the probability distributions as points in an n-dimensional Euclidian space, the limit points of the set of distributions obtained by successive applications of the transformation to some fixed initial distribution are restricted to those which have their total probability attached only to the two extreme points—the largest and the smallest. Further it is shown that the number of limit points is not more than two and if there are two limit points they are the transforms of each other. Thus, the sequence usually is " oscillatorily convergent."

This result offers the theoretical foundation for an acceleration procedure of the optimum gradient method proposed by Forsythe and Motzkin in their paper " Asymptotic properties of the optimum gradient method " [*Bull. Amer. Math. Soc.* (1951) **57**, 183]. A min-max proof for the least upper bound of the convergence rate which was first obtained by Kantrovitch is given and further we can see that if the matrix is ill-conditioned, then without acceleration procedure the convergence rate tends near to its worst possible value as the approximation proceeds. Some numerical examples are given by using the same data as those treated by Forsythe in " Punched-card experiments with accelerated gradient methods for linear equations" [*Nat. Bur. Stand., Applied Mathematics* Series No. 39 (1954), 55-69] and good agreement is noted between these examples and the theoretical observations.

(H. Akaike)

BARTON, D. E. & DAVID, FLORENCE N. (University College, London) **1.3** (4.6)

Contagious occupancy—*In English*

J. R. Statist. Soc. B (1959) **21**, 120-133 (23 refs., 4 tables, 3 figs.)

The problem of the distribution of elements into boxes (cells, groups, categories, compartments, etc.) is one of some antiquity, dating back to De Moivre. The authors offer a generalised approach to the problem by which known results to the problem can be attained and offer also a new contagious-type distribution. Assume N boxes and n balls dropped randomly. Let N Poisson variables $t_1, t_2, \ldots, t_N$ characterise the numbers in each of the N boxes. From the joint probability law of the t's, with n fixed the Vth factorial moment of the distribution of k, the number of boxes which contain no ball can be deduced and hence the probability law of k. This distribution is that found by De Moivre. If instead of a Poisson variable a positive binomial variable is postulated, this is equivalent to putting an upper bound to the capacity of the box. The distribution of k under this set-up is that as previously deduced by Romanovsky. In assuming, as a third possibility, that the distribution within a cell is that of a negative binomial, the authors are introducing the idea of contagion as first proposed by Lüders. The distribution of the number of empty boxes is new. Finally, the p of the positive binomial is allowed to vary uniformly $o<p<1$ and the resulting distribution of the number of empty boxes is found to be that derived by Freund and Posner.

It is shown that the effect of assuming the positive (or negative) binomial for the individual cell distribution is such as to make the "zero" distribution very different from the classical Poisson result. Nevertheless, provided the ratio n/N is fixed all the zero distributions discussed tend rapidly to that of the normal with increasing N. Using the normal approximations of the classical distribution of zeros for the distribution of the criterion under the null hypothesis, and the contagion distribution for the distribution of the criterion under the alternative hypothesis, the appropriate number of boxes (relative to the number of balls) is discussed. All other things being equal, it is decided that the number of boxes should be as large as possible.

The number of zeros when the individual cell distribution is a positive binomial is suggested as a criterion for analysing the randomness of points on a lattice. If the alternative hypothesis is that of "fluctuation" occupancy it is suggested that the lattice should be so divided as to make the number of boxes equal the number of balls.

Finally, the zero criterion is compared with the index of dispersion.

(Florence N. David)

345

BARTON, D. E. & DAVID, FLORENCE N. (University College, London) **1.3** (5.6)

The dispersion of a number of species—*In English*

J. R. Statist. Soc. B (1959) **21**, 190-194 (1 ref., 1 table)

The problem of the distribution of various species of plants in a field (divided into equal areas) is discussed in terms of balls of various colours distributed into N boxes.

It is shown that the number of boxes containing balls of both of two colours is distributed hypergeometrically (conditional on given numbers of boxes with at least one of a given colour) when the different colours are distributed independently. This is generalised for k colours. Four models previously treated by the authors [*J. R. Statist. Soc.* B (1959) **21**, 120-133] are considered as generating the occupancy distribution for a given colour: (*a*) random, (*b*) not more than d balls per box, (*c*) contagious occupancy (balls tending to fall together) and (*d*) with a varying degree of density of balls from box to box. The probability distribution function of the number of boxes with both colours is given under (*a*). The number of different colours represented in a box is then considered, and its mean and variance given for models (*a*), (*b*) and (*c*). It is shown to be approximately a binomial variable (exactly so when the overall numbers of each colour are the same), and a numerical example shows graduation by the binomial probability distribution function to be satisfactory.

A generalised (k-colour) form of model (*b*) is discussed where the total of all colours in a box is not more than m. If the total of all balls in a given box is less than m by n_0, then it may be regarded as containing n_0 blanks and $n_1, \ldots n_n$ of the different colours. The results may be arranged as an $N\times(k+1)$ contingency table. The usual joint distribution of frequencies is shown to hold so that the standard χ^2 test for independence of the different colours may be used. As m becomes large this is shown to be equivalent to the sum of the separate Poisson indices of dispersion for each colour. The implication is that this is the appropriate test for independent distribution of colours under model (*a*).

(D. E. Barton)

Editorial Note: see abstract No. 345, **1.3.**

346

BHARUCHA-REID, A. T. (University of Oregon, Eugene) **1.5** (1.6)

On the convergence of a sequence of generalised random variables in an Orlicz-space—*In German*

Bull. Acad. Polon. Sci. III (1959) **7**, 425-427

The paper contains three theorems which give the necessary and sufficient conditions for the convergence of a sequence of generalised random variables in a separable Orlicz-space to a generalised random variable in the three following senses:

(i) weakly sequential almost surely,
(ii) weakly almost surely,
(iii) strongly almost surely.

(L. Kubik)

BHARUCHA-REID, A. T. (University of Oregon, Eugene) **1.0** (10.0)

On random operator equations in Banach spaces—*In English*

Bull. Acad. Polon. Sci. III (1959) **7**, 561-564

In this note the following random operator equation is considered:

$$(T(\omega)-\lambda I)x(u, \omega) = y(u, \omega)$$

where $x(u, \omega)$ and $y(u, \omega)$ are generalised random variables with values in a Banach space; $T(\omega)$ is a random operator and I is the identity operator. This equation arises in the study of certain linear functional equations in probabilistic functional analysis, especially in the theory of stochastic boundary value problems.

The author studies the existence and measurability of the resolvent operator associated with a random operator $T(\omega)$.

(L. Kubik)

BRAUMANN, P. (Faculty of Sciences, Lisbon) **1.1** (–.–)

Symmetric infinitely devisible probability law—*In German*

Rev. Fac. Ciencias Lisboa, A (1959) **7**, 255-262

A law is called indefinitely divisible if its characteristic function is, for every n, the characteristic function of the sum of n identically distributed random variables.

For the characteristic functions of the indefinitely divisible random variables we can give some representations —of Levy-Khintchine, Levy and Kolmogoroff.

The author defines the symmetry of an indefinitely divisible law and obtains for each representation the necessary and sufficient conditions for the symmetry of an indefinitely divisible law.

(J. T. de Oliveira)

CIUCU, G. (Bucharest University) **1.5** (10.1)

An exceptional case for the limit distribution function of sums of random variables of a stationary continuous Markoff process with a finite set of states—*In Rumanian*

Gaz. mat. şi fiz. A (1958) **3**, 141-142 (2 refs.)

The asymptotic behaviour is investigated of the distribution function of sums of random variables forming a stationary continuous Markoff process with a finite set of states. The functional equations verified by the characteristic functions of finite sums are given, the limit characteristic function being determined as the limit of these characteristic functions corresponding to finite sums. Moreover, the associated system of equations with partial derivations is deduced. By making use of certain boundary conditions and of certain results given previously by Onicescu [Analele Univ. C. I. Parhon—Bucureşti, Şeria St Naturii (1956) **12**, 9-12]. The limit distribution function of the normally distributed sums is obtained as a sum of a Gaussian distribution function and of an error function of an oscillating type.

(G. Sîmboan)

DANTZIG, the late D. VAN. (Mathematical Centre, Amsterdam) **1.8** (10.4)
On some questions in mathematical decision theory—*In French*
Report SP 69—*Statist. Dept., Math. Centre, Amsterdam* (1959) 22 pp. (27 refs.)

Two applications and a critical survey of mathematical decision theory are given. The first example concerns the optimal height of the dykes of the Netherlands and the second a new approach by De Leve to solve a large class of generalised inventory problems.

The author criticises the notions of subjective probability and utility. The following example is given to show that maximising expected utility can be a very ineffective way of optomising.

In a game, the player gains 2 with probability $q = 1-p$, and loses q/p with probability p, both 2 and q/p being expressed in differences of utilities. The expected value of the gain equals q. Suppose the player can choose p arbitrarily under the condition that the amount to be paid, if he loses, will have to be deposited before playing. So F being his capital, maximising expected utility entails maximising q under the condition $q/p \leqq F$. The optimal value of p equals $1/(F+1)$ and losing means ruin. If this game is played a large number of times, it can be shown that ruin will certainly occur in a finite number of plays. Moreover, a come-back is impossible. If expected utility is *not* maximised, a lot of strategies are available, which never imply ruin and give a reasonable chance of gain.

It is pointed out that the use of Von Neumann and Morgenstern's theory of coalitions does not seem to be very fruitful for solving problems in modern politics because too many unrealistic assumptions have to be made.

Notwithstanding all criticisms, the author regrets the knowledge and the experience of mathematicians not being used more directly when really important decisions have to be taken.

(J. Kriens)

FISZ, M. (Math. Inst., Polish Academy of Sciences, Warsaw) **1.5** (–.–)
On necessary and sufficient conditions for the validity of the strong law of large numbers expressed in terms of moments—*In English*
Bull. Acad. Polon. Sci., III (1959) **7**, 221-225

The following theorems are proved:

Theorem 1. There do not exist for sequences of independent, symmetric random variables satisfying relations $|X_i| < i$ $(i = 1, 2, \ldots)$ necessary and sufficient conditions for the validity of the strong law of large numbers expressed in terms of the variances σ_i^2 of X_i.

Theorem 2. Let r be an arbitrary given natural number. There do not exist for arbitrary sequences $\{X_i\}$ $(i = 1, 2, \ldots)$ of independent random variables necessary and sufficient conditions for the validity of the strong law of large numbers expressed in terms of the moments $m_{i1}, \ldots, m_{ir}$, where $m_{ij} = EX_i^j$.

(L. Kubik)

FRÉCHET, M. (University of Paris) **1.6** (–.–)

On the practical importance of " almost certain convergence "—*In French*
Rev. Int. Statist. Inst. (1958) **26**, 26-28 (2 refs.)

It has been asserted that of the two notions " almost certain convergence " and " convergence in probability ", only the second one is of interest in applied statistics. The author raises some objections against this opinion: since almost certain convergence implies convergence in probability, the first notion cannot be less useful than the second one. Moreover, it is shown that almost certain convergence provides applied statisticians with more complete information than convergence in probability.

(M. Fréchet)

353

GILBERT, E. N. (Bell Telephone Labs., Murray Hill, N. J.) **1.3** (–.–)

Random graphs—*In English*
Ann. Math. Statist. (1959) **30**, 1141-1144 (4 refs., 1 table)

The following two paragraphs are a direct quotation from the introduction to the paper being abstracted:—

" Let N points, numbered 1, 2, ..., N, be given. There are $N(N-1)/2$ lines which can be drawn joining pairs of these points. Choosing a subset of these lines to draw, one obtains a graph; there are $2^{N(N-1)/2}$ possible graphs in total. Pick one of these graphs by the following random process. For all pairs of points make random choices, independent of each other, whether or not to join the points of the pair by a line. Let the common probability of joining be p. Equivalently, one may erase lines, with common probability $q = 1-p$ from the complete graph.

" In the random graph so constructed, one says that point i is connected to point j if some of the lines of the graph form a path from i to j. If i is connected to j for every pair i, j, then the graph is said to be connected. The probability P_N that the graph is connected, and also the probability R_N that two specific points . . . are connected, will both be found."

A generating series for P_N is derived on the basis of a result stated in a previous paper. From this an explicit representation of P_N is derived, as well as a recurrence relation

$$1-P_N = \sum_{k=1}^{N-1} \binom{N-1}{k-1} P_k q^{k(N-k)}, \quad P_1 = 1,$$

which is easier to use for large N than the explicit formula. Similarly, the recurrence relation for R_N takes the form

$$1-R_N = \sum_{k=1}^{N-1} \binom{N-2}{k-1} P_k q^{k(N-k)}.$$

A small table of values of P_N and R_N for several values of q is presented.

For a sufficiently large value of N so that the above recurrence relations are unwieldly, it is shown that

$$P_N = 1-Nq^{N-1}+0(N^2q^{3N/2})$$
$$R_N = 1-2q^{N-1}+0(Nq^{3N/2})$$

(R. M. Durstine)

354

HARRIS, B. (Stanford University, California) **1.2** (2.4)

Determining bounds on integrals with applications to cataloging problems—*In English*

Ann. Math. Statist. (1959) **30**, 521-548 (8 refs., 9 tables)

We assume that a random sample of size N has been drawn from a multinomial population with an unknown and perhaps countably infinite number of classes. The experimenter wishes to predict $d(\alpha)$, the number of classes that will be observed in a second sample of size αN, $\alpha > 1$, (or when the sample size is increased by $(\alpha-1)N$ additional observations); and $C(\alpha)$, the coverage of a second sample (or augmented sample), where $C(\alpha) = \Sigma p_j$, the sum is to be taken over those classes for which at least one representative has been observed in the sample.

It is shown that

$$E\{d(\alpha)\} \sim E(d) + E(n_1)E^*\{1/x - 1/x \exp(\alpha-1)x\}$$

and

$$E\{C(\alpha)\} \sim 1 - E(n_1)/N - \{E(n_1)/N\}E^*\{1/x - 1/x \exp(\alpha-1)x\}$$

where d is the number of classes observed, n_1 is the number of classes occurring once in the sample, and E^* denotes an expected value taken with respect to a distribution function unknown to the experimenter since it depends on all the unknown parameters $\{p_j\}$. It can, however, be shown that the moments of this distribution function are approximately given by certain functions of the expected values of n_r, the number of cells occurring r times in the sample. Hence, if we replace the expected values of n_r by the observed values, we have estimates of the moments of this distribution. For sufficiently large N, we may suppose for fixed k, that the first k moments are known. Thus, if in $E\{d,(\alpha)\}$ and $E\{C(\alpha)\}$ we replace the expected values of d and n_1 by their observed values, then, we need only determine the remaining expected values.

A reasonable procedure is to determine upper and lower predictors of $d(\alpha)$ and $C(\alpha)$ by determining the suprema and infima of expected values subject to moment constraints.

The author proceeds to show that if $g(x)$ is a continuous, bounded, and monotonic function on $[0, b]$; and its first k derivatives exist and are monotonic in $(0, b)$, and $g(x)$ is not linearly dependent on $1, x, x^2, \ldots, x^k$, then we can easily determine the supremum and infimum of the expected value of $g(x)$, where the extremisation is over all distribution functions on $[0, b]$, whose first k moments are specified. It is shown that the two extremising distribution functions are uniquely determined, one being continuous at b, and one has a saltus at b. The extremising distribution functions are characterised and the case $b \to \infty$ is also considered. The explicit solutions for $k = 0, 1, 2, 3$ are computed and applied to the specific problem at hand.

The results in the paper are compared with formulations of the same problem by Fisher *et al.*, Good, and Goodman, and several numerical examples are given.

Several related results are also obtained and low order moments of d, n_r, and C are computed. The large sample exponential approximations for $E(d)$, $E(C)$, and $E(n_r)$ are developed in appendices.

(B. Harris)

355

IHM, P. (Botanic Institute, Freiburg, Germany) **1.0** (0.3)

Numerical evaluation of certain multivariate normal integrals—*In English*

Sankhyā (1959) **21**, 363-366 (6 refs., 1 fig.)

A method of integrating the multivariate normal density over rectangular parallelopipeds is presented. The problem is reduced to the evaluation of a single integral. The integrand is a product of a number of integrals of the univariate normal density over one dimensional intervals and a normal density. Such a reduction is possible whenever the covariance matrix $\boldsymbol{A}$ has the form $\boldsymbol{A} = \boldsymbol{D} + c^{-2}ii'$, i being the unit vector $(1, 1, \ldots, 1)$, $\boldsymbol{D}$ a diagonal matrix with positive elements in the diagonal cells. The final integration has to be done by numerical quadrature. The author suggests that by a certain transformation of the variable of integration, results of sufficient accuracy for all ordinary purposes can be obtained with the help of a planimeter.

(S. John)

ISII, K. (Inst. Statist. Math., Tokyo) **1.5** (1.2)

On a method for generalisations of Tchebycheff's inequality—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **10**, 65-88 (11 refs., 7 figs.)

Numerous authors have contributed to the generalisation of Tchebycheff's classical inequality. In this paper, however, a sufficiently general new theory is developed upon a unified method of obtaining probability inequalities.

Suppose that the first $2n$ moments $\mu_1, \ldots, \mu_{2n}$ of a one-dimensional random variable X are given, and let E be any closed (or open) set without any such restrictions as "semi-infinite" or "symmetrical about the origin". The best possible upper and lower bounds of $P_r\{X \in E\}$ are given in terms of $\mu_1, \ldots, \mu_{2n}$. For example, under certain conditions the upper bound is the supremum of linear forms of $\mu_0 = 1$, $\mu_1, \ldots, \mu_{2n}$ with the same coefficients as the polynomials which dominate the indicator $\chi_E(x)$ of the set E.

When E is a closed set, the upper extremal distribution has the following properties: either (i) the extremal distribution has its first $2n$ moments equal to $\mu_1, \ldots, \mu_{2n}$ and its spectrum consists of at least $n+1$ points and at most $3n$ points, or (ii) the extremal distribution has its first $2n-1$ moments equal to $\mu_1, \ldots, \mu_{2n-1}$ and the $2n$th moment less than μ_{2n}, and the spectrum consists of at least n points and at most $3n-3$ points. When E is an open set, the statement about the lower extremal distribution is true. The results are fairly general and cover the classical results of Tchebycheff-Markoff-Stieltjes and that of Wald. Proofs of the statements are essentially reduced to the extension problem of positive linear functionals on Banach spaces.

In the last section the author considers three typical examples: a discrete set, an interval symmetrical about the mean, and two-dimensional cases.

In Example 1, the necessary and sufficient condition that a discrete distribution on the point set $\{\xi_1, \ldots, \xi_{2n}\}$ has the moments $\mu_1, \ldots, \mu_{2n}$ is given in terms of the two sets of values.

In Example 2, all possible cases of upper bounds are shown with the character of the extreme distributions.

In Example 3, two-dimensional distributions with the given means, variances and covariance are considered. Regions, on which the probability measure is evaluated, are restricted to two-dimensional open intervals with the length proportional to the standard deviation of the corresponding axis. The mean point of the distribution is assumed to be on one of the two diagonals of the regions. The sharp lower bounds of the probability are given in three cases according to the position of the mean point.

(M. Sibuya)

357

LAHA, R. G. & LUKACS, E. (The Catholic Univ. of America, Washington, D.C.) **1.2** (2.0)

On distributions whose moments are majorated by moments of a known distribution—*In English*

J. Aust. Math. Soc. (1959) **1**, 113-115 (2 refs.)

Let $\alpha_k^{(1)}$, $\alpha_k^{(2)}$ denote the kth moments about $x = 0$, and $f_1(t)$, $f_2(t)$ the characteristic functions, of two distribution functions $F_1(x)$, $F_2(x)$. Suppose that $f_2(t)$ is an analytic function of t, and that for any k, $|\alpha_k^{(1)}| \leqq |\alpha_k^{(2)}|$. Using well-known properties of analytic functions, the authors prove that the circle of convergence of the Maclaurin series for f_1 contains the circle of convergence for f_2, and that in the case of f_2 being an entire function f_1 is also an entire function with order not exceeding the order of f_2. The method of proof is simpler than that used by Milicer-Gruzewska (1949) for a particular case—essentially $F_2(x)$ being normal with zero mean. An example is given showing that the strip of convergence (parallel to the imaginary axis in the complex t-plane) of the integral defining $f_1(t)$ may not contain that of $f_2(t)$.

(E. W. Bowen)

358

ONICESCU, O. (Bucharest University) **1.0** (0.0)

Notes on Boolean algebras—*In French*

Rev. Math. Pures Appl. (1959) **4**, 345-350 (6 refs.)

This paper is divided into three parts. In the first part the structure of a probabilised Boolean algebra A is investigated. The notions of atom and quasi-atom are introduced by making use of indecomposability respectively of the elements of the Boolean algebra and of their corresponding measures. Further, considering the Boolean algebras of atoms and quasi-atoms and the so-called continuous kernel, a canonical decomposition of A is given, as for the corresponding measure. Moreover, it is shown that the continuous kernel is characterised by the Darboux property.

In the second part it is proved that any part of A has a maximal element.

Starting from a denumerable additive Boolean algebra A, in the last part the author suggests a method for constructing a punctual space and a Borelian field isomorphic to A.

(R. Theodorescu)

PRATT, J. W. (Harvard University) **1.6** (1.5)

On a general concept of " in probability "—*In English*

Ann. Math. Statist. (1959) **30**, 549-556 (6 refs.)

Mann and Wold have given a general theory and notation for stochastic limits and limit distributions [*Ann. Math. Statist.* (1943) **14**, 217-226]. In so doing they gave a probability meaning to the O and o notation that has proved so useful in analysis. In the present paper the definition of " in probability " is broadened and clarification is attempted by separating the stochastic and limiting aspects of the problem.

Thus, given sequences of random variable $\{X_n\}$ and numbers $\{r_n\}$, the usual definition of o_p is as follows: $X_n = o_p(r_n)$ if for every positive ϵ, η there exists N such that for $n > N$ $P_r[x_n/r_n \leqq \eta] \geqq 1-\epsilon$. In words the probability that x_n/r_n is small tends to unity as n increases.

The two aspects of the problem are separated if the definition reads $X_n = o_p(r_n)$ if for every positive ϵ there exists c_n with $P_r \mid X_n \mid > 1-\epsilon$ and $c_n = o(r_n)$.

This form of the definition is generalised in terms of sets in the product space of the random variables, and it is shown that usual theorems on interchangeability of functional operations and stochastic limits are true and in fact are almost trivial. Some examples are given and some remarks made about possible further generalisations.

(D. G. Chapman)

SANKARANARAYANAN, G. (Annamalai University, Madras) **1.5** (–.–)
A note on the equi-distribution of sums of independent random variables—*In English*
Indian Math. Soc. J. (1958) **22**, 93-98 (1 ref.)

If $X_1, X_2, \ldots$ is a sequence of independent random variables with a common distribution function and $S_n = X_1 + \ldots + X_n$ then Robbins has proved that the class H of functions $h(x)$, such that with probability one

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h(S_j) = M(h)$$

where $M(h)$ is a suitably defined mean, contains all uniformly almost periodic functions. The author proves the same result when S_j is defined as

$$X_{p_1+p_2+\ldots+p_{j-1}+1} + \ldots + X_{p_1+\ldots p_j}$$

where $p_1 < p_2 < \ldots$ is a sequence of integers increasing to infinity.

(K. R. Parthasarathy)

361

THEODORESCU, R. (Institute of Mathematics, Bucharest) **1.1** (1.6)
On an abstract integral in probability—*In English*
Rev. Math. Pures Appl. (1958) **3**, 277-281 (5 refs.)

Let us consider a random vector X, i.e. an application of an interval U into the set of all random elements (in the sense of Mourier), defined on a fixed probability space and with values in a given Banach space. A random vector K is said to be weakly integrable in probability if every linear functional of X is integrable in probability on all subintervals U_0 of U—and if there is a random element $\tilde{X}$ corresponding to each subinterval U_0 of U such that the integral in probability on U_0 of every linear functional of X is equal to the random variable formed by applying this linear functional to $\tilde{X}$. By definition, the random element X corresponding to U_0 is the weak integral in probability on U_0 of X. This integral is uniquely defined almost everywhere with respect to the probability measure occurring in the basic probability space. Several properties of this integral are then given, in analogy to those of the Pettis- and the Fan-integral.

(G. Sîmboan)

362

TRYBULA, S. (Math. Inst., Polish Academy of Sciences, Wroclaw) **1.0** (–.–)
On lower bounds for probability moments—*In English*
Bull. Acad. Polon. Sci. III (1959) **7**, 267-269

Suppose that the random variable X has an absolutely continuous distribution with density function $f(x)$. The nth probability moment π_n is defined by $\pi_n = \int_{-\infty}^{\infty} f^n(x)dx$. In this note the author gives the greatest lower bound of parameter π_n for random variables with finite kth central absolute moment.

(L. Kubik)

ADAM, A. (Institüt für Statistik, Universität Wien) **2.1** (10.5)
Measures and indices of distributions—*In German*
Metrika (1959) **2**, 239-244 (4 refs.)

The paper deals with a measure of distribution and indicates some of its properties. Let x, y, z be some attribute vectors; $p(x, y, z)$ the corresponding density function. The measure of distribution is defined by

$$P(z \mid y, x) = \exp\{-E \ln p(z \mid y, x) \geqq 0.$$

The connection with the entropy of information theory is obvious. The author shows that the properties of P are very similar to corresponding properties of probabilities. In the case of a normally distributed one-dimensional variable x, $P(x) = \sqrt{2\pi e}\sigma(x)$; an analogous formula holds in the case of a multivariate normal distribution. For all continuous distributions, the inequality $0 \leqq P(x) \leqq \sqrt{2\pi e}\sigma(x)$ is mentioned. According to the definition of P, the variance has the meaning of a set of informations.

The author then defines an index of distributions R by means of P:

$$R(z \cap y \mid x) = P(z \mid y, x)/P(z \mid x).$$

He interprets R as the reduction of variance for the vectors y and z respectively if, and provided the influence of x is eliminated, some dependence exists between y and z. For continuous distribution functions the result

$$R(x \cap y) \leqq R(Ax \cap y)$$

holds for non-singular linear transformations A. Some other properties of R are mentioned in the paper, especially an analogy to Bayes' rule. R is considered as a central measure of information.

(J. Roppert)

AGGARWAL, O. P. & GUTTMAN, I. (Purdue Univ., Indiana and McGill Univ., Quebec) **2.8** (11.1)
Tables of the cumulative distribution functions of samples from symmetrically truncated normal distributions—*In English*
Ann. Inst. Statist. Math., Tokyo (1959) **11**, 55-68 (1 ref., 5 tables)

Tables of the functions $\Phi_i(t)$ which are the cumulative distribution functions of the sample mean minus one-half unit, are given to five significant figures for samples of size i from distributions obtained by truncating at $\pm a$ the normal distribution with zero-mean and unit variance. Tables are given for $i = 1, 2, 3, 4$ for values of $a = 1(0{\cdot}5)3$ with t advancing in steps of 0·01 until the value of a.

The density functions of these distributions were given, and these tables used, in the paper " Truncation and tests of hypothesis," [*Ann. Math. Statist.* (1959) **30**, 230-238] by the same authors, in the course of examining the loss of power when using tests based on the assumption that the variable has a complete normal distribution when, in fact, the distribution is " truncated ".

(H. Akaike)

Editorial Note: see also abstract No. 187, **2.8.**

AITCHISON, T. (Dept. of Mathematics, University of Glasgow) 2.8 (2.6)

A statistical theory of remnants—*In English*

J. R. Statist. Soc. B (1959) **21**, 158-168 (5 refs., 3 tables)

The probability distribution function $g(y)$ of the fractional part $y = x-[x]$ of a random variable x is studied as a basis for the theory of remnants. This probability distribution function is the sum of $f(y+r)$ over all integral values of r: y is nearly uniform (rectangular) in the interval [0, 1). The upper bound (d) of $|g(y)-1|$ is studied as is $\delta = |\mathscr{E}(y)-\frac{1}{2}|$, where $2\delta \leqq d$. Using the Poisson summation formula—that $g(y)$ is the sum over r of $\phi(2\pi r)\exp(-ry2\pi i)$ where ϕ is the cumulative function of x—it is shown that $\delta \leqq d/2x$. Moreover, if σ, μ are the scale and location parameters and T is $(x-\mu)/\sigma$, $\sigma(d+1)$ is the sum over r of $f_T(r/\sigma)$ where f_T is the probability distribution function of T. Both of these results assume that ϕ is real valued and non-negative.

When x is normally distributed, computation shows that $2\pi\delta \doteqdot d$ whilst $d \doteqdot \exp(-2\pi^2\sigma^2)$ when $\sigma > \frac{1}{2}$: the Sheppard correction result. Since this implies that $\mathscr{E}(y)$ is hardly affected by μ or σ and no control of these can thus affect $\mathscr{E}(y)$, attention is given to the case where σ is small. It is shown that $\mathscr{E}(y)$ has two stationary values, a minimum at $\lambda_0 > \frac{1}{4}$ and a maximum at $1-\lambda_0 > \frac{3}{4}$ where $\lambda_0 = \mu-[\mu]$, i.e., the fractional part of μ.

Computation of λ_0 is discussed and its value with corresponding values of $\mathscr{E}(y)$ and $Var(y)$ are tabled: they are found to increase from zero at $\sigma = 0$ to nearly rectangular values for $\sigma > 0{\cdot}4$ (Table 2). It follows that a small decrease (of $2\lambda_0$) in μ can lead to a large increase in $\mathscr{E}(y)$—from minimum to maximum. Two tests based upon a sample of remnants are suggested for the hypothesis that a process producing articles of average length of an integer plus λ_0 is under control. The first uses the approximately normal probability distribution function of $\bar{y}$, and the second, dispensing with measurement, gauges the remnants against $2\lambda_0$. If r pieces are found to be less than $2\lambda_0$ the normal approximation to the binomial is used. The second test is shown to be more powerful (Table 3).

Other probability distribution functions of x are discussed e.g., Cauchy, log-normal and Pearson Type III, in so far as they entail modifications. Finally, the paper shows briefly how the theory may be extended to k-dimensional remnants: when r is multinomial the probability distribution function is obtained as an infinite series analogous to the univariate case.

(D. E. Barton)

367

AMATO, V. (University of Catania) 2.5 (2.7)

Approximate relationships between the negative binomial and the Γ distribution—*In Italian*

Industria (1959) 3-16 (11 refs., 4 tables)

An approximate relation between the negative binomial distribution and the gamma distribution is given whose use is justified when the parameters of the negative binomial assumes moderately high values. The author, starting from the agreement of the theoretical models to the empirical distributions, demonstrates that in the negative binomial when one of the two parameters is allowed to diverge, its limit is the gamma distribution. This relation is very useful since the values of the gamma distribution can easily be found in Pearson's tables and the negative binomial or Poisson's compound distribution is particularly suitable to describe situations where the average can vary from case to case.

(P. Bandettini)

BODMER, W. F. (Dept. of Genetics, Cambridge, England) **2.9** (0.4)

The limiting frequencies of integers with a given partitional characteristic—*In English*
J. R. Statist. Soc. B (1959) **21**, 134-143 (8 refs.)

Any integer may be factorised uniquely as a product of powers of distinct prime factors. If a given integer has for factors a primes to the $(\alpha+1)$th power, b primes to the $(\beta+1)$th and so on (with $\alpha<\beta>\ldots>\eta\geqq 1$)—the rest being unit powers, then it is said to have the partitional characteristic $Q\equiv(\alpha^a\beta^b\ldots\eta^n)$. Rényi [*Acad. Serbe. Sci., Publ. Inst. Math.* (1955) **8**, 157-163] has shown that the proportion of integers up to x with a given Q tends to a fixed limit $N(Q)$ as $x\to\infty$ and that $1/6\pi^2N(Q)$ is the sum (over all sets of ω different primes) of $f_\alpha(p_1)x\ldots xf_\eta(p_\omega)$ where ω is the weight of Q and $f_\alpha(p)=1/(p+1)p^{a-1}$. For all the primes up to an indefinitely large number, this is a symmetric function of them and $N(Q)$ is evaluated by expressing this as a sum of products of the generalisation of the power sums (i.e. the single sums of the products of the f's taken all at the same argument) and replacing these by infinite sums over all primes. In fact any such sum is of the form $U_{rs}=\Sigma p^{-r}(1+p)^{-s}$ and these rapidly convergent series are the basic elements computed.

The coefficients in the relations between the symmetric functions are bipartitional functions which are simple multiples of those of David and Kendall's Table I (the symmetric functions here are non-augmented) and much of this paper is devoted to proving Sukhatme's rules for their evaluation.

Those values of $N(Q)$ which are greater than zero on attaining five decimal places, are tabulated and compared with the observed frequencies in the integers 8446 to 8775. The proportion T_{rs} (of integers with $\alpha=r$, $a=s$) is also computed as is the proportion F_r (with $\alpha=r$) for the values of α contributing to the table of $N(Q)$. Further, Rényi's numbers d_ω (the sum of $N(Q)$ for a given ω). Some relative frequencies conditional in a and α are given and Yule's measure of association is computed giving, for instance, the association between integers divisible by third and fourth prime powers. The T_{rs} are expressed as a symmetric function (as is $N(Q)$ which is evaluated in terms of single sums V_{rs} which are sums over all primes of $(p-1)/(p^{r+1}-p)^s$. This enables an independent check to be made on the totals of the $N(Q)$'s. Likewise the F_r are the differences of the Riemann Zeta-function and so check the total of the T_{rs}.

(D. E. Barton)

BURKHOLDER, D. L. (University of Illinois, Urbana, U.S.A.) **2.0** (4.6)

On the existence of a best approximation of one distribution by another of a given type—*In English*
Ann. Math. Statist. (1959) **30**, 738-742. (1 ref.)

Let $F(x)$ and $G(x)$ be two distribution functions and (a, b) an ordered pair of real numbers with $a>0$. The author investigates the problem of finding a pair (a, b) for which the distribution function $F(ax+b)$ of the type of $F(x)$ is close to $G(x)$. The approximation of the distribution function and the optimal pair (a_0, b_0) are defined by postulating that

$$\sup\,|\,F(a_0x+b_0)-G(x)\,|\,=M$$

where $M=\inf\sup\,|\,F(ax+b)-G(x)\,|$. In these expressions the supremum is to be taken over all real x while the infimum is taken over all positive a and all real b.

The author gives a simple example which shows that an optimal pair (a_0, b_0) does not always exist. He derives then two sufficient conditions (Theorems 1 and 2) for the existence of an optimal pair.

Theorem 1. Let A (with respect to B) be the set of all points such that $1/3\leqq F(x)\leqq 2/3$ (with respect to $1/3\leqq G(x)\leqq 2/3$). If both sets A and B are non-degenerate, an optimal pair (with $a_0>0$), satisfying the above postulate exists.

Theorem 2. If $F(x)$ is continuous while $G(x)$ is a step function which has at most a finite number of discontinuity points in each finite interval, then an optimal pair (a_0, b_0) exists.

Theorem 3 states that if the set of optimal pairs is non-empty then it is a convex set. This is used to describe the set of optimal pairs and to obtain a condition for the existence of a unique, optimal pair (a_0, b_0).

The results can be applied to the problem of estimating a scale and location parameter by the minimum distance method.

(E. Lukacs)

CERTAINE, J. (Nuclear Devel. Corp. of America, White Plains, N.Y.) **2.6** (11.5)
On sequences of pseudo-random numbers of maximal length—*In English*
J. Ass. Comp. Mach. (1958) **5**, 353-356 (3 refs., 1 table)

A widely used method for generating pseudo-random numbers on electronic computers consists of calculating successive powers (modulo m) of a number a. The number m is usually taken to be the power of the base of the system which is used to represent numbers to the computer; in general, m will be the largest power of 2 or 10 which will fit into the machine.

This paper gives the length of the largest non-repeating sequence that is possible for any given m and gives a method for determining the number a which is required to yield this largest sequence. Table 1 contains generating numbers for some of the more widely used computing machines, including desk calculators.

(D. Teichroew)

371

COX, D. R. & BRANDWOOD, L. (Birkbeck College, London) **2.8** (1.8)
On a discriminatory problem connected with the works of Plato—*In English*
J. R. Statist. Soc. B (1959) **21**, 195-200 (5 refs., 3 tables)

Plato wrote five lesser works between the *Republic* and the *Laws* and this paper attempts to order them in time by means of the frequencies of the thirty-two different patterns of the last five (long or short) syllables of each sentence. The above mentioned major works differ markedly in respect to these patterns and each provide some 4000 examples so that their proportions are taken as being population values, the proportions in the lesser works being alone subject to random fluctuations (these yielding from 150 to 950 sentences). General theory in respect of k characters (here the thirty-two patterns) is worked out, the frequencies being multinomially distributed with probabilities $\{\theta_{ri}\}$, $r = 0, 1;\ i = 1, \ldots k$; in the extremal populations $\{\pi_r\}$. If $\{f_i\}$ are the relative frequencies in a lesser work, the discriminant is $\bar{s} = \Sigma f_i s_i$, where $s_i = \log(\theta_{1i}/\theta_{0i})$.

A model for the probabilities $\{\theta\lambda_i\}$ is an intermediate population π_λ is proposed ($0 \leqq \lambda \leqq 1$) and this has $\bar{s}$ as sufficient statistic for λ or, equally, for its expected value $\phi(\lambda)$, shown to be monotonic in λ. The estimated standard error of $\bar{s}$ is computed for each of the five lesser works, each based on its own sample. The sentence endings are supposed to be statistically independent and each work homogeneous. The latter assumption is supported by internal comparisons not given in this paper between chapters in the *Republic* and the *Laws*. The final ordering is *Republic*, *Timaeus*, *Sophist*, *Critias*, *Politicus*, *Philebus*, and *Laws* with various degrees of significance for the successive paired comparisons. It is pointed out that the ordering attained is not in accord with the views of the majority of classical scholars, although a minority group have reached similar conclusions by independent argument.

(D. E. Barton)

372

FUCHS, A. (Strasbourg University, France) **2.1** (4.5)
On the inequality of Slutsky—*In French*
Publ. Inst. Statist. Paris (1959) **8**, 161-163 (1 ref.)

In this short paper the author gives a simple proof of the Slutsky inequality (between moments) and compares it with other similar inequalities—one being deduced from the well-known relationship between the arithmetic and geometric averages.

(Mlle. Gervaise)

HEUMANN, H. & NOBIS, E. (Kabelwerk Rheydt Akt.-Ges., Rheydt, Rheinland) **2.3** (11.3)
Calculation of the average and economic consumption of thermoplastic material for insulating cables and conducting wires—*In German*
Metrika (1959) **2**, 230-238 (4 refs., 1 table, 3 figs.)

Most electric cables and metallic conducters are covered by insulating thermoplastic material. This insulating cover has to be of a certain defined minimum thickness: cores of a lower thickness must be rejected. In a manufacturing process carried out by extruders, variation of the wall thickness produced cannot be avoided. In a great number of measurements the thickness could be found to be normally distributed. For every machine and process the standard deviation of the thickness has a known value, which is unaffected by changes of the mean value. Adjusting the extruder to an average wall thickness that is smaller or only insignificantly greater than the specified limit results in the material consumption being high. This is because plenty of cores are not sufficiently thick and must be rejected. They will be burned and prepared for a new process in order to save the valuable metallic wire. If the average wall thickness produced is significantly greater than the specified limit, the consumption of insulating material will also become undesirably high. In this case, the wall thickness of most cores is greater than needed.

This paper describes a method for calculating that value of the mean which minimises the consumption and loss of material. In addition to the two terms representing the losses caused by too small or too great differences between the mean and the specified limit, the equation for all losses contains a third term as an equivalent for the costs of burning and preparing the rejected cores. They are independent of the diameter of the wires but proportional to the number of cores. The total loss function is minimised. The result is found by a graphical solution of a transcendental equation. It is believed that the method discussed in this paper could be successfully applied to other industries.

(H. Schauerte)

JOHNSON, N. L. (University College, London) **2.5** (3.3)

On an extension of the connection between Poisson and χ^2 distributions—*In English*
Biometrika (1959) **46**, 352-363 (11 refs., 3 tables)

An extension, connecting the non-central χ^2 distribution with the distribution of the difference between two independent Poisson variables, is discussed with the purpose of approximating the latter distribution by an approximation of the non-central χ^2 distribution. A similar approximation to the difference between a negative binomial variable and an independent Poisson variable is found by an approximation of the non-central F distribution.

In the case of the non-central χ^2 distribution with even degrees of freedom, the Poisson probability may be evaluated using a Bessel function. If neither of the Poisson parameters is small, Johnson proposes using a normal distribution to approximate the distribution of the difference of the Poisson variables: he later modifies this for skewness. The accuracy of the approximation is expected to increase with the non-centrality parameter λ, Patnaik's first non-central χ^2 approximation [*Biometrika* (1949) **36**, 202-232] being more accurate when λ is small. This is shown in a table for comparison of known approximations to the upper and lower 5 per cent. points of the non-central χ^2 distribution. The type of variation, however, is not quite as anticipated. Patnaik's approximation to the normal distribution is compared with Johnson's, and when the upper limit of the probability integral is large, Johnson's approximation gives values of this limit larger than Patnaik's.

The non-central F distribution is treated in a similar way and an analogue of the normal approximation to the non-central χ^2 is given with some modification for skewness.

A short discussion is given of the two cases (Poisson variable)—(negative binomial variable) and (negative binomial variable)—(negative binomial variable) and their correspondence to the component of variance model in analysis of variance. The distribution of F'', where F'' is the ratio of two non-central χ^2 distributions divided by their degrees of freedom, is given as a mixture of negative binomial distributions.

The remainder of the paper concerns the applications of the concept of a "mixture" to non-central χ^2, negative binomial and non-central F distributions.

(Barbara Saw)

Editorial Note: see also abstract No. 387, **3.2.**
Pearson

KAO, J. H. K. (Cornell University, Ithaca, N.Y.) **2.5** (4.3)

The mixed Weibull distribution in life-testing of electron tubes—*In English*
Nat. Conv. Trans., Amer. Soc. Qual. Contr. (1959) 267-289 (8 refs., 2 tables, 8 figs.)

In broad terms electron tube failures can be classified into two types: (i) catastrophic or sudden failures and (ii) wear-out or delayed failures. A tube which is operating normally and then suddenly becomes inoperative is defined as a catastrophic failure. Since, in general, these failures are of a mechanical nature, they may appear as soon as the tube is put into operation; the failure rate due to catastrophic failures decreases with time. A tube whose quality characteristics have gradually fallen below given specifications is defined as a wear-out failure. Such failures are generally of a chemical or electrical nature. For most applications, very few wear-out failures occur until some particular time period has elapsed, after which the failure rate increases with time.

For the time to failure, x, the proposed distribution functions for catastrophic and wear-out failures are, respectively,

$$F_1(x) = \begin{cases} 1-e^{-x^{\beta_1}/\alpha_1} & \text{for } x \geqq 0 \\ 0 & \text{for } x \leqq 0 \end{cases}$$

$$F_2(x) = \begin{cases} 1-e^{-(x-\gamma_2)^{\beta_2}/\alpha_2} & \text{for } x \geqq \gamma_2 \\ 0 & \text{for } x \leqq \gamma_2 \end{cases}$$

where $\alpha_1, \alpha_2 > 0$, $0 < \beta_1 < 1$, $\beta_2 > 1$, and $\gamma_2 \geqq 0$. The distribution function of the life time of electron tubes is

$$F(x) = pF_1(x) + (1-p)F_2(x)$$

where p is the proportion of catastrophic failures.

The failure rate, $z(x)$ is defined as

$$z(x) = F'(x)/[1-F(x)].$$

The mixed distribution function, $F(x)$ is useful in describing the frequently observed situation of an initial high failure rate, followed by a decreasing rate and then an increasing rate. The initial failures are attributable to catastrophic failures and the later failures to wear-out.

Graphical methods are given for estimating the parameters α_1, β_1, α_2, β_2, γ_2 and p. The construction of a confidence band for $F(x)$ is also described.

(D. W. Gaylor)

KIMBALL, A. W. & LEACH, E. (Oak Ridge National Lab., Tennessee) **2.2** (0.4)
Approximate linearisation of the incomplete β-function—*In English*
Biometrika (1959) **46**, 214-218 (7 refs., 1 table)

Karl Pearson pointed out that the sum of a number of binomial terms is equal to the incomplete beta-function ratio. The authors are concerned with a quantity M, the probability of death from a dose D of radiation where $M = I_x(p, q)$ and x is a function of D. Pearson's nomenclature, which the authors retain, is unfortunate in this particular connection since x is equal to the p of the common binomial parlance. The authors require an appropriate solution of the equation $M = \alpha$. They remark that, from observations on M and D, regression estimates can be obtained, but that the use of $I_x(p, q)$ as an expected value presents grave difficulties. They sought, therefore, for a transformation which would give a linear relationship between functions of the dependent and independent variables.

Scheffé and Tukey gave an approximate solution to $M = \alpha$ in terms of the α percentage point of the χ^2 distribution with $2q$ degrees of freedom. Since this approximation cannot be expected to serve over the whole range of x, the authors give their modification of it. (An excellent approximation due to Tukey did not seem appropriate since the regression could not be made linear). The Wilson-Hilferty cube-root transformation for $\chi^2/2q$ is applied to the χ^2 of the Scheffé-Tukey transformation. The authors accordingly finish with an approximation to the incomplete beta-function ratio which is an incomplete integral of a unit normal curve. The upper bound of the integral is cumbersome but the results are close to the exact values as may be seen from the tables.

Tukey's approximation is also given in the table. It never appears worse than the one suggested by the authors and is often better. A simplification of the authors own approximation is also given.

(Florence N. David)

LAHA, R. G. (Catholic University of America, Washington) **2.1** (2.3)
On the laws of Cauchy and Gauss—*In English*
Ann. Math. Statist. (1959) **30**, 1165-1174 (8 refs.)

Let x and y be two independently distributed normal random variables with the zero mean and common variance. It is then known that the quotient x/y is distributed according to Cauchy's law (with median zero). It is of interest to investigate whether the converse statement holds, since this could yield tests of normality. However, the converse is in general not true. Counter examples were given by the author [*Proc. Nat. Acad. Sci. U.S.A.* (1958) **44**, 222-223], by Steck [*Ann. Math. Statist.* (1958) **29**, 310-312], and by Mauldon [*Quart. J. Math.* (1956) **7**, 155-160].

In the present paper the author proves first the following theorem:

Theorem 1. Let x and y be two independently and identically distributed random variables with common distribution function $F(x)$. Suppose that the quotient $w = x/y$ has a Cauchy distribution with zero median. Then $F(x)$ is symmetric about $x = 0$ and is absolutely continuous with a density function $f(x) = F'(x) > 0$. Moreover, x has an unbounded range and $f(x)$ satisfies for all u the relation

$$\int_0^\infty f(x)\, f(ux)x\, dx = c_0/(1+u^2) \text{ where } c_0 \text{ is a constant.}$$

The author gives then a characterisation of the normal distribution.

Theorem 2. Suppose that the assumptions of Theorem 1, and also the following two conditions are satisfied

(i) $F(x)$ has finite moments of all order,

(ii) $\zeta(z) = \exp[iz \ln | x |]$ has no zeros in its region of regularity (z complex).

Then $F(x)$ is a normal distribution.

The author has continued this study in a paper "On a class of distribution functions where the quotient follows the Cauchy law" [*Trans. Amer. Math. Soc.* (1959) **93**, 105-215].

(E. Lukacs)

LAUBSCHER, N. F. (National Physical Research Lab., Pretoria) **2.2** (6.2)

A note on Fisher's transformation of the correlation coefficient—*In English*

J. R. Statist. Soc. B (1959) **21**, 409-410 (2 refs.)

Fisher's $\tanh^{-1}$ transformation for normalising the distribution of the correlation coefficient, r, is well known. Other transformations have been suggested, notably perhaps the $\sin^{-1}$ transformation by Harley. The author considers whether, instead of taking $\log\{(1+r)/(1-r)\}$, it will be possible to obtain a transformed variance independent of ρ by taking $\log\{(c_1+r)/(c_2-r)\}$ and choosing c_1 and c_2 appropriately. He investigates this by the usual moment expansion of a Taylor series and concludes that $c_1 = c_2 = 1$ in Fisher's transformation is the optimum for that transformation.

(Florence N. David)

MULLER, M. E. (Princeton University) **2.3** (11.5)

A comparison of methods for generating normal deviates on digital computers—*In English*

J. Ass. Comp. Mach. (1959) **6**, 376-383 (19 refs., 2 tables)

A number of procedures have been suggested for transforming one or more uniform variates into a normal deviate on a digital computer. This paper describes the following six procedures.

(i) *Direct Approach.* Starting with two independent uniform variates, U_1 and U_2, on the interval (0, 1) let

$$X_1 = (-2 \ln U_1)^{1/2}\mathrm{Cos}\, 2\pi U_2;$$
$$X_2 = (-2 \ln U_1)^{1/2}\mathrm{Sin}\, 2\pi U_2.$$

Then X_1 and X_2 are normally and independently distributed (0, 1).

(ii) *Inverse Approach.* Solve $X = X(U)$ in the equation

$$U = 1/\sqrt{2\pi}\int_{-\infty}^{x} e^{-t^2/2}\,dt,\ 0 \leqq U \leqq 1.$$

Over most of the interval (0, 1) for U, $X = X(U)$ is approximated by Tchebysheff-type polynomials. As X increases, the degree of the polynomial is increased. For $127/128 \leqq U \leqq 1$, a truncated continued fraction expansion is used. This method was previously described by the author [*J. Ass. Comp. Mach.* (1958) **12**, 167-174].

(iii) *Central Limit Approach.* $\theta = \sum_{i=1}^{n} U_i$ approaches normality. This is studied for $n = 12$.

(iv) *Rejection Approach.* Consider U_1 and U_2 as above and $-b \leqq X \leqq b$. Compute $Y = -2b^2(U_1 - 0{\cdot}5)^2$. If $\ln U_2 \leqq Y$, $X = b(2U_1 - 1)$ is used as a normal deviate; otherwise reject the pair.

(v) *Hastings' Approach.*

$$X = X^*(q) = \eta - \left[\sum_{i=0}^{2} a_i\eta^i \Big/ \sum_{i=0}^{3} b_i\eta^i\right]$$

where $\eta^2 = -2 \ln q$ and $q = 1 - U[X(q)]$, as in (2); $0 < q \leqq 0{\cdot}5$. Values of a_i and b_i are given.

(vi) *Teichroew's Approach.* θ is used, except the range is restricted; for $n = 12$, $2 \leqq \theta < 10$, where

$$\mathrm{Prob}[2 \leqq \theta \leqq 10] \geqq 1 - 2(10)^{-5}.$$

For $n = 12$, let $r = (\theta - 6)/4$. Then

$$X = \sum_{i=0}^{4} a_{2j+1} r^{2j+1}.$$

The values of a are given for this case. They are based on a fitted Tchebysheff polynomial: the details are given in a doctoral thesis at North Carolina State College.

A brief critique is given of each approach. Comparisons were made on an IBM 704 on the basis of computing time per deviate, precision and memory space required. The inverse approach is much faster and the rejection approach much slower than the other four approaches. The Central Limit and Teichroew approaches require the least amount of memory space and the direct and inverse approaches by far the most. The direct and rejection methods are the most precise and the Hastings and inverse approach are least precise; Teichroew's method is not good for deviates on the extreme tail.

(R. L. Anderson)

WILKS, S. S. (Princeton University)

Recurrence of extreme observations—*In English*
J. Aust. Math. Soc. (1959) **1**, 106-112.

This paper considers independent observations being drawn from a population with continuous distribution function $F(x)$. Let y be the rth largest in the first m observations, and n the number of further observations required for k of these further observations to exceed y. By elementary methods involving the known distribution of $F(y)$, the negative binomial distribution of n given $F(y)$, and integration over $F(y)$, it is shown that the probability function of n is given by

$$p(n) = m\binom{n-1}{k-1}\binom{m-1}{r-1}\binom{m+n-1}{k+r-1}^{-1}(m+n)^{-1}, \qquad n = k, k+1, \ldots,$$

independently of the functional form of $F(x)$. The mean of this distribution is $mk(r-1)^{-1}$; in the case of $k = 1$, its variance is $mr(m-r+1)(r-1)^{-2}(r-1)^{-1}$. The limiting distribution for large values of m of the ratio $z = n/m$ is determined; if $k = 1$, this distribution has density function $r/(1+z)^{r+1}$, $z>0$.

The same $p(n)$ with $r = r_1+r_2(<m)$ gives the probability that n further observations will be required for k of these further observations to lie outside the interval between the r_1th from smallest and the r_2th from largest of the first m observations.

Some numerical results are given in connection with predicting exceedances of the extremes obtained in a preliminary sample; for example, if m is large, the probability is approximately 0·95 that fewer than $3{\cdot}47m$ further observations will be required for an observation to lie outside the interval between the smallest and largest of the first m observations; in the same situation, the probability is approximately 0·95 that more than $m/38$ observations will be required.

(E. W. Bowen)

EEDEN, CONSTANCE VAN. (Mathematical Centre, Amsterdam) **3.2** (11.1)

Some approximations to the percentage points of the non-central t-distribution—*In English*
Statist. Dept., Math. Centre (1960), Report S242.

In this paper the following approximations to the percentage points of the non-central t-distribution are considered:

(i) Three closely related approximations based on a normal approximation to the distribution of $z+k\sqrt{\chi^2}$, where z possesses a $N(0, 1)$-distribution and χ^2 a χ^2-distribution with f degrees of freedom. Two of these approximations were derived by Jennet & Welch [*Suppl. J. R. Statist. Soc.* (1940) **7**, 80-88] and by Johnson & Welch [*Biometrika* (1940) **31**, 362-389] respectively. The third one was derived by Hendricks [*Ann. Math. Statist.* (1936) **7**, 210-221] for the percentage points of the central t-distribution and could be generalised for the non-central t-distribution.

(ii) Four closely related approximations based on a method used by Goldberg & Levine [*Ann. Math. Statist.* (1946), **17**, 216-225] for the central t-distribution. In this case the percentage points are expressed in a power series in $1/f$, the coefficients being functions of a $N(0, 1)$-variable.

(iii) The approximation of Merrington & E. S. Pearson [*Biometrika* (1958) **45**, 484-491] based on a Pearson type IV—approximation to the non-central t-distribution.

(iv) The approximation of Harley [*Biometrika* (1957) **44**, 219-224] based on an approximation of the distribution of a function of t by the distribution of the correlation coefficient in samples from a bivariate normal distribution.

(v) An approximation described in Owen's table of factors for one-sided tolerance limits for a normal distribution (Sandia Corporation Monograph, 1958). This approximation is based on an (approximate) relation between one- and two-sided tolerance limits.

(vi) An approximation based on a normal approximation to the non-central t-distribution.

These approximations are compared numerically with the exact percentage points for several values of the non-centrality parameters δ, the probability α and the number of degrees of freedom f. Several tables are given to make this comparison possible. For the central t-distribution the approximations are especially important for large values of f, where tables are not available. Then one of the approximations (ii) is the best one: it is identical with the exact percentage points in three decimal places for $0{\cdot}50 \leqq \alpha \leqq 0{\cdot}99$. For the non-central t-distribution the values $f = 2(1)9$ were used. For these values of f the approximations (i) and (ii) are the most important ones. In some cases (iii) or (iv) may be better, but for these approximations more computational work is needed. The approximations (v) and (vi) are not recommended; (vi) gives only reasonable results for the central t-distribution.

Further, the report contains a description of the exact tables of the non-central t-distribution.

383

(Constance van Eeden)

HARTER, H. L. (Wright Air Development Center, Dayton, Ohio) **3.8** (4.3)

The use of sample quasi-ranges in estimating population standard deviation—*In English*
Ann. Math. Statist. (1959) **30**, 980-999 (21 refs., 6 tables, 1 fig.)

The use of sample quasi-ranges in estimating population standard deviation is discussed, and a number of tables are provided for cases of interest. Section 1 deals with the normal population, and Section 2 with rectangular and exponential populations.

The rth quasi-range, w_r, of a sample of size n is defined as the range of the $(n-2r)$ sample values remaining after removal of the r smallest and r largest; thus w_0 stands for the range of the whole sample.

Section 1 which deals with the normal population $N(\mu, 1)$, begins with a tabulation of $E(w_r)$ in Table 1. This table gives $E(w_r)$, accurate to within a unit in the sixth decimal place, for $r = 0(1)8$ and $n = (2r+2)(1)100$. Table 2 presents var (w_r), accurate to within a unit in the fifth decimal place, for $r = 0(1)8$ and $n = (2r+2)(1)100$.

The second table is concerned with efficiency of estimates, expressed in terms of $\hat{\sigma} = s/c_2$, where s is the sample standard deviation and c_2 is the correction factor to provide unbiasedness. The efficiency of an estimate $\tilde{\sigma}$ is given by $\text{eff}\,\tilde{\sigma} = \text{var}\,\hat{\sigma}/\text{var}\,\tilde{\sigma}$. Table 3 shows the efficiency of the unbiased estimate $w_r/E(w_r)$ for $r = 0(1)8$ and $n = (2r+2)(2)50(5)100$, accurate to within 0·01 per cent. Table 4 compares for each $n = 4(1)100$, the efficiencies of the most efficient estimate based on one sample quasi-range, the most efficient estimate based on linear combinations of two adjacent quasi-ranges, and the most efficient estimates based on two quasi-ranges w_r and $w_{r'}$ with $r < r' \leqq 8$; each estimate is given, along with its efficiency. Comparisons with Grubbs-Weaver estimates are made in Section 1.6.

Section 2 deals with non-normal populations. For the standard (zero mean and unit variance) rectangular population, the unbiased estimate of σ based on the range is given in Table 5 for $n = 2(1)100$; the numerical coefficients on w_0 are accurate to within a unit in the sixth decimal place.

For the standard (unit mean and variance) exponential population the most efficient estimates (in comparison with the efficient estimate $\bar{x}$) based on quasi-ranges were found to have efficiencies varying from 50·00 per cent. to 61·73 per cent., and so were not tabulated. It is stated that investigation of efficiencies of estimates based on single order statistics has begun.

Finally, Section 2.3 considers the bias involved if normality is assumed when the population actually is exponential or rectangular. Table 6 shows these biases, for $n = 2(1)100$, for the various estimates tabulated in Table 4.

(R. Hooke)

384

KUIPER, N. H. (Landbouwhogeschool, Wageningen) **3.0** (2.0)

On the random cumulative frequency function—*In English*

Proc. Kon. Ned. Akad., Wetensch. A (1960) **63** *Indag. Math.* (1960) **22**, 32-37 (6 refs., 2 figs.)

Two random variables are called " isomorous " if they have the same cumulative frequency function. If the independent mutually isomorous random variables $x_1, \ldots, x_n$ are isomorous to a given continuous random variable y, with cumulative distribution function $F(x)$, and if $F_W(x)$ is the cumulation frequency function of the random sequence $W = (x_1, \ldots, x_n)$, then

Theorem 1. (Gnedenko-Mihalavicz) $u = \int_{F_W(x) > F(x)} dF(x)$ is a uniform random variable on [0, 1], and

Theorem 2. (Birnbaum, Pyke). The optimal value x_{opt} of the random function $F_W(x) - F(x)$ (that is the [random] value x for which the minimum is obtained) is such that $F(x_{\text{opt}}) = U$ is a uniform random variable on [0, 1].

In this paper the author gives a simple proof of these two theorems, which is based on two ideas:

(*a*) The results are independent of the continuous random variable y (hence independent of the function $F(t)$. Then he restricts to the case that y is uniform on [0, 1].

(*b*) The interval [0, 1] can be mapped into the reals modulo one, on which operates the natural group of translations. In terms of the reals modulo 1, theorems 1 and 2 are then deduced from the invariance properties under this group.

(N. Kuiper)

PACHARES, J. (Hughes Aircraft Co., Culver City, California) **3.3** (11.1)

Table of the upper 10 per cent. points of the studentised range—*In English*

Biometrika (1959 **46**, 461-466 (4 refs., 3 tables)

Let w be the sample size of n values sampled from a normal population, and s be a sample estimate of the standard deviation based on ν degrees of freedom and distributed independently of w. The author gives to three significant figures, a new table of the upper 10 per cent. points of the studentised range, $q = w/s$ for $n = 2(1)20$ and $\nu = 1(1)20$, 24, 30, 40, 60, 120, ∞. Details are given of the formulae used and of the method of computation which was carried put on a IBM 704 computer.

For checking purposes the author recomputed the upper 5 per cent. points already given in Table 29 of Pearson & Hartley's *Biometrika Tables for Statisticians* **1,** (1954) and discovered several minor errors in their table. In addition the editors of *Biometrika* were able to correct some similar minor errors in their table of the upper 1 per cent. points from some highly accurate, but as yet unpublished, calculations by Harter. The corrected tables of both the upper 5 per cent. and 1 per cent. points, given to two decimal places, and for the same values of n and ν, are therefore included with the author's new table of the upper 10 per cent. points.

(D. Harris)

PEARSON, E. S. (University College, London) **3.2** (2.1)

Note on an approximation to the distribution of non-central χ^2—*In English*
Biometrika (1959) **46**, 364.

In connection with Johnson's paper [*Biometrika* (1959) **46**, 352-363], this note briefly sets out some details of another approximation to the distribution of non-central χ^2. This particular approximation used a χ^2 distribution with the correct mean, standard deviation and skewness factor (β_1) but not necessarily exactly the correct starting place. These conditions appear suitable when the curve does not rise sharply at the lower end.

The errors between the exact values and upper and lower 5 per cent. points of the approximation are given in column (v) of Table 1 in Johnson's paper. The conclusion is that " Taken as a whole, it is clear that for the particular values of the parameters selected, the three-moment χ^2 approximation is better than the other approximations ". It is emphasised by the author that the search for accurate approximations to non-central statistics was not the main purpose of Johnson's paper.

(Barbara Saw)

Editorial Note: see also abstract No. 375, **2.5.**
Johnson

PILLAI, K. C. S. (Statistical Center, University of the Philippines) **3.3** (11.1)

Upper percentage points of the extreme studentised deviate from the sample mean—
In English
Biometrika (1959) **46**, 473-474 (4 refs., 1 table)

Let $x_1 \leqq x_2 \leqq \ldots \leqq x_n$ be an ordered sample of size n from a normal population with variance σ^2, the mean of the sample $\bar{x}$, and s_ν the square root of an independent mean square estimate of σ^2 based on ν degrees of freedom. Then the extreme studentised deviate from the sample mean is defined as $t_n = (x_n - \bar{x})/s_\nu$ or $(\bar{x} - x_1)/s_\nu$. The upper 5 per cent. and 1 per cent. points of t_n are given in a new table for $n = 2(1)\,10, 12$ and $\nu = 1(1)\,10$. The method of preparation of this table is discussed.

(D. Harris)

Editorial Note: see also abstract No. 389, **3.3.**
Pillai & Tienzo

PILLAI, K. C. S. & TIENZO, B. P. (Statistical Center, University of the Philippines) **3.3** (3.8)

On the distribution of the extreme studentised deviate from the sample mean—*In English* *Biometrika* (1959) **46**, 467-472 (11 refs.)

Let $x_1 \leqq x_2 \leqq \ldots \leqq x_n$ be an ordered sample of size n from a normal population with variance σ^2. The sample mean is $\bar{x}$ and s_ν is the square root of an independent mean square estimate of σ^2 based on ν degrees of freedom.

Let $u = (x_n - x)/\sigma$ or $(\bar{x} - x_1)/\sigma$

$t_n = (x_n - x)/s_\nu$ or $(\bar{x} - x_1)/s_\nu$.

The distribution of u has previously been derived by several authors [e.g. David, H. A., *Biometrika* (1956) **43**, 449] and probability integrals given for n ranging from 2 to 25. Percentage points of t_n, the studentised extreme deviate from the sample mean, have also been obtained for sample sizes $n = 3(1)\ 10$, 12 and some values of $\nu \geqq 10$.

The present paper develops the distributions of u and of t in series form, for $n = 3$, 4, 5 and $\nu \leqq 10$. Percentage points of t_n derived from formulae given in this paper are incorporated in the more extensive tables of the 5 per cent. and 1 per cent. points given by Pillai [*Biometrika* (1959) **46**, 473-474].

The authors point out that percentage points for $\nu < 10$ are particularly useful in many situations, for example in experiments with small numbers of treatments. This is illustrated by an example of a randomised block experiment with five treatments and three blocks, in which the test is applied to the difference between the largest treatment mean and the general mean.

(D. Harris.)

Editorial Note: see also abstract No. 388, **3.3.**

RYBARZ, J. (Technische Hochschule, Vienna) **3.1** (2.4)

An elementary proof of the theorem on which the χ^2 method is based—*In German* *Metrika* (1959) **2**, 89-93 (3 refs.)

For the theorem that, given a multinomial distribution with k classes, the chi-square statistic follows asymptotically the chi-square distribution with $k-1$ degrees of freedom, the proof of K. Pearson [*Phil. Mag* (1901) **50**, 157-172] and the proof using the characteristic function [Cramér, *Mathematical Methods of Statistics* (1945)] are well known.

Morgenstern [*Metrika* (1958) **1**, 239-241] gives a more elementary proof based on complete induction from k to $k+1$. For $k = 2$ he uses the asymptotic normality of the binomial distribution. For $k+1$ he splits up the statistics into three parts. The first follows the chi-square distribution with $k-1$ degrees of freedom, the second is asymptotically normal and the third vanishes asymptotically.

This paper gives another approach using the asymptotic normality of the Poisson distribution. The conditioned joint distribution of k random variables is the multinomial distribution if each of k random variables follows a Poisson distribution with the parameter np_i ($\Sigma p_i = 1$) subject to the restriction that the sum of the k variables is n. The distribution of each random variable, and also the sum of the k variables lacking the restriction about the sum, converge with increasing n to a normal distribution. The conditioned asymptotic distribution, again subject to the restriction that the sum is n, is a $k-1$ dimensional normal distribution: this leads to the theorem.

It should be mentioned that the same idea is stated by Sir Ronald Fisher [*J. R. Statist. Soc.* (1922) **85**, 87-94] and is also considered in the expository paper by Cochran [*Ann. Math. Statist.* (1953) **24**, 315-345].

(E. Walter)

WATSON, G. S. (Australian National University, Canberra) **3.1** (5.1.)
Some recent results in chi-square goodness-of-fit tests—*In English*
Biometrics (1959) **15**, 440-468 (25 refs., 1 table)

The author begins by considering the effect of estimation of parameters on the distribution of Pearson's X^2 criterion of goodness of fit. Frequently the estimates are obtained by maximum likelihood or some asymptotically equivalent method applied to the data before the grouping necessary for the chi-square test is made. The author points out that this leads to a distribution which, even asymptotically, is not χ^2 but differs from χ^2 with the usual degrees of freedom by a positive definite quadratic form involving the unknown parameters. Hence, even though the correct estimators (i.e., best asymptotically normal estimators based on cell frequencies) may be troublesome to obtain, the author recommends their use and suggests that they may be found by an iterative solution of the appropriate (e.g., maximum likelihood) equations (with the simpler estimators as starting values) with less difficulty than that required for an assessment of the effects of using alternative estimators.

The disturbing effect of incorrect estimation can be partitioned out, an operation described as being equivalent to calculating X^2 with the estimators which result from one cycle of iteration to the maximum-likelihood equations. This appears to be satisfactory since, in large samples, one iteration should be sufficient.

These results, which are stated in the introductory section, are demonstrated in Section 2 of the paper, as well as the relation of the effect of estimation on chi-square and its partitions to Neyman's smooth goodness-of-fit tests. It is stated that the theory of Section 2 is adequate to deal with

(*a*) true multinomial distributions,
(*b*) grouped discrete distributions, and
(*c*) continuous distributions grouped by using intervals which are fixed for all samples.

In the third section, finite samples from the multinomial are discussed. It is shown that, as the sample size increases, X^2 tends to the chi-square distribution obtained unconditionally in the limit. A table is given to make a comparison of three X^2 statistics for small sample situations. Continuous distributions are treated in Section 4. An argument is presented to justify the intuitive procedure in large sample situations of ignoring the manner in which the class intervals were found, as long as the method of their construction is " fair ". The final section of the paper treats two general methods for partitioning X^2.

The author's conclusion is that " . . . The theory and application of X^2 to true multinomial situations leaves little to be desired. For continuous distributions, the approach is less natural although . . . it may be made to solve the main problems satisfactorily."

(R. H. Brown)

391

WEIR, J. B. de V. (University of Glasgow) **3.1** (–.–)
Standardised *t*—*In English*
Nature (1960) **185**, 558.

Students t may be standardised by expressing it in units of its own standard deviation; thus, $t_s = t\sqrt{\{(\nu-2)/\nu\}}$, where ν is the number of degrees of freedom. The procedure reduces the variability of the tails with ν for a given probability P, and in particular for $P = 0{\cdot}05$ and $\infty \geqq \nu \geqq 4$, t_s lies between 1·96 and 2·0 whereas t lies between 1·96 and 2·776. The finding extends the applicability of certain " large " sample methods down to 4 degrees of freedom.

The standardising is extremely simple to apply—merely divide the error sum of squares by $\nu-2$ instead of by ν and use this value to calculate the appropriate standard error (SE). For example, if d is the difference between two means and $d \geqq 2$ SE, $P \leqq 0{\cdot}05$, that is the difference is significant at the 5 per cent. level. If $d < 1{\cdot}96$ SE, $P > 0{\cdot}05$ and the difference is not significant. Between these two limits of d, $P \simeq 0{\cdot}05$; actually $0{\cdot}0535 > P > 0{\cdot}0455$. Again, ± 2 SE will give a close approximation to 95 per cent. confidence limits (actually between 95 and 95·45 per cent.). When the formula is applied to the correlation coefficient, r, we find that r is significant at the 5 per cent. point if $r \geqq 2/\sqrt{(\nu+2)}$.

(J. B. de V. Weir)

392

ZIA-UD-DIN, M. (Punjab University, Lahore) **3.0** (2.1)

The expression of k-statistic k_{11} in terms of power sums and sample moments—*In English*

Ann. Math. Statist. (1959) **30**, 825-828 (7 refs.)

With the help of tables by Abdel-Aty [*Biometrika* (1954) **41**, 253-263] the author gives in this note the expression for the k-statistic k_{11} in terms of the power sums

$$s_r = \sum_{i=1}^{n} a^r.$$

In a previous note [*Ann. Math. Statist.* (1954) **25**, 800-803] the author found the corresponding expression for k_9 and k_{10}.

(R. Hooke)

AZORIN, F. (Central University of Venezuela, Caracas) **4.3** (6.1)
Statistical considerations on the Cobb-Douglas function—*In Spanish*
Estadística Española (1959) **4**, 19-29 (9 refs.)

This paper is concerned with the so called Cobb-Douglas production function. Not with its general interpretation or justification but with the estimation of parameters. After a brief introduction the author considers the fitting of an hyper-surface to the exponential function in which k independent variables represent inputs, whereas the dependent variable y is to be taken as output.

Supposing a simple random sample of size n in each of $k+1$ groups, in which each group represents a variable or productive element plus a product quantity, the observations being made in particular firms or industries. The parameters of the fitted exponential function have to be estimated.

For the sake of simplicity, the case is first considered as being of one dimension. The two parameters are estimated by the least squares method, the result being maximum likelihood estimates in the general linear hypothesis. For practical reasons the fitting of an hyper-plane is made directly to the logarithms of the observations.

Now, special attention is given to characteristics which can be defined as the marginal productivity, obtained from the previously estimated parameters. There would no be difficulty if the variable x and the estimated y might be considered as constants. But $\hat{y}$ is really an estimator which takes different values in the possible samples of size n. A method due to Carter and Hartley consists in an approximation for the calculation of the variance of the estimated marginal productivity. Extending the results to the k-dimensional case one has a matrix expression of the above variance.

However, the estimator of the variance of the marginal productivity is not unbiased. In order to obtain an estimation of that bias, the author observes that it can be considered as a product of a constant into $b\hat{y}(x)$.

Now the mathematical expectation of the above expression can be written as a sum of two terms, the second of which may be expanded in a series of linear functions of moments. Finally, dividing by x, an expression of the bias of the marginal productivity is obtained in terms of x, y, the logarithms of the observations, the population regression coefficients and the variance of the logarithm of the production. A first approximation is obtained when the mathematical expectation of the estimation of y is approximately equal to y.

When the distribution of the logarithm of the estimated production is normal, the moments of odd order vanish and a simpler expression of the relative bias is given.

It is further shown that the bias is not negligible relative to the standard error of the marginal productivity. Finally, it is pointed out how to increase the accuracy by decreasing the bias and not only through a large precision of the estimate.

(S. Rios)

CLEMANS, K. G. (U.S. Naval Ordnance Test Station, China Lake, California) **4.4** (2.5)
Confidence limits in the case of the geometric distribution—*In English*
Biometrika (1959) **46**, 260-264 (4 refs., 2 tables, 2 charts)

The author is concerned with a problem in life-testing. X is the number of trials and m is its expected number. X has a distribution called here the geometric distribution although it is sometimes also called the sequential Poisson or the sequential binomial. Suppose N articles each life-tested with the ith item requiring X_i trials before failure. A sufficient estimator for m is the mean of the x's. Also the sum of the X's is distributed as a negative binomial. Thus for a given m and N, limits can be found for $\bar{X}$, and consequently for m for given N and $\bar{X}$.

The charts were computed on a IBM computer and the author remarks that the programme will be kept available. The charts are in agreement with the work of Anscombe if the extra corrective terms are used where necessary. Examples are given and interpolation and extrapolation into the charts are discussed.

(Florence N. David)

CROW, E. L. & GARDNER, R. S. (Nat. Bur. Stand., Colorado and U.S. Naval Ord. Test Sta., China Lake) **4.4** (11.1)

Confidence intervals for the expectation of a Poisson variable—*In English*
Biometrika (1959) **46**, 441-453, (14 refs., 4 tables, 2 figs.)

A table of optimum two-sided confidence intervals is given for confidence coefficients 0·80, 0·90, 0·95, 0·99 and 0·999 and all values of the variable from 0 to 300. For a given confidence coefficient, the confidence belt thus tabulated is determined by the properties:

(i) that of all confidence belts for the Poisson expectation which are based directly on the Poisson variable c, it is as narrow as possible in the c-direction,
(ii) that of all such equally narrow confidence belts it has the smallest possible upper confidence limits.

These give a system of the same type as that tabulated by Crow [*Biometrika* (1956) **43**, 437-432] for the parameter of the binomial distribution. Reference should be made to this earlier paper for full details of definitions and calculations. A property of Poisson probability sums which is discussed in the present paper resulted in simplification of the calculations, most of which were carried out on a high-speed electronic computer.

Denote by δ_4 the system described above, by δ_1 the system of central confidence intervals given by Pearson and Hartley [*Biometrika Tables for Statisticians* **1**, (1954)], by δ_3 the system having the property (i) above, defined by a condition due to Sterne [*Biometrika* (1954) **41**, 275-278] and by δ_5 the shortest unbiased central system (obtained by adding a random variable, uniformly distributed over (0, 1), to the observed discrete variable).

These systems are compared, and the percentage decreases in length of interval achieved by using δ_4 instead of δ_1, is shown to be appreciable for at least small values of c. Expected lengths of 90 per cent. confidence intervals for $0 \leqq m \leqq 14$ were calculated and, although those of δ_5 were the smallest in the range, those of δ_4 exceeded them by 0·06 at most for $m \geqq 6$ and were shorter than those of δ_1 by 0·45-0·9 over the full range. Comparing δ_1, δ_3 and δ_4 with various large sample approximations shows that it is a useful and valid procedure to employ δ_4 intervals as far as they are tabulated together with a given large sample approximation for larger c due to Hald [*Statistical Theory with Engineering Applications* (1952) New York: Wiley].

Thus the δ_4 system is believed to be preferable to δ_1 in many applications, though the author does not propose its use for one-sided confidence intervals since division of probabilities between the tails of the distribution is not restricted.

(D. Harris)

GRAYBILL, F. A. & DEAL, R. B. (Oklahoma State University, Stillwater) **4.1** (7.3)

Combining unbiased estimators—*In English*
Biometrics (1959) **15**, 543-550. (3 refs.)

The problem considered is that of combining linearly two independent unbiased estimators $\hat{\alpha}_1$, $\hat{\alpha}_2$ of a parameter α when the ratio of the variances σ_1^2, σ_2^2 of $\hat{\alpha}_1$, $\hat{\alpha}_2$ is unknown. The combined estimator $\hat{\alpha}$ is termed by the authors a "uniformly better unbiased estimator" of α (than $\hat{\alpha}_1$, $\hat{\alpha}_2$) if its variance is less than or equal to the smaller of σ_1^2, σ_2^2 for all values of σ_1^2, σ_2^2. It is well known that no linear combination with constant weights can be uniformly better but under certain circumstances it is possible to find random weights which achieve this.

An important special case is studied and the following main result obtained. Let x, y be normal variates with common mean μ and unknown variances σ_1^2/n_1, σ_2^2/n_2; let $m_1 s_1^2/\sigma_1^2$, $m_2 s_2^2/\sigma_2^2$ be distributed as χ^2 with m_1, m_2 degrees of freedom and suppose all random variables are independent. Then a necessary and sufficient condition that

$$\hat{\mu} = (n_1 s_2^2 x + n_2 s_1^2 y)/(n_1 s_2^2 + n_2 s_1^2)$$

is an unbiased estimator of μ which is uniformly better than either x or y is that m_1 and m_2 are both larger than 9.

An application of the theorem is to the recovery of interblock information in a balanced incomplete block design. Suppose it is desired to estimate the difference of two treatment effects. The best linear combination $\hat{\mu}$ of intra-block and inter-block estimates is known (e.g. Kempthorne, *Design and Analysis of Experiments*, p. 535). From the theorem it follows that $\hat{\mu}$ is uniformly better than either individual estimate if (i) $rt-b-t+1 \geqq 18$ and $b-t = 9$ or if (ii) $b-t \geqq 10$ where r = no. of replications of each treatment, b = no. of blocks and t = no. of treatments. So if (i) or (ii) is true, inter-block information should always be used. It should be noted that Yate's method of combining inter-block and intra-block estimates is different. This method is discussed by Graybill & Weeks [*Ann. Math. Statist.* (1959) **30**, 799-805].

The inaccuracy due to estimating the weights in the balanced incomplete block is also studied by comparing the variance of $\hat{\mu}$ with that of the estimator appropriate when the ratio σ_1^2/σ_2^2 is known.

(H. A. David)

GUTTMAN, I. (McGill University, Montreal) **4.5** (2.5)

Optimum tolerance regions and power when sampling from some non-normal universes—*In English*

Ann. Math. Statist. (1959) **30**, 926-938 (3 refs., 8 tables)

This paper presents optimum β-expectation tolerance regions for the single and double exponential distributions and exhibits their corresponding power functions. A tolerance region is a β-expectation tolerance region if the average probability content of the region is at most β. It is a similar β-expectation tolerance region if the average probability content of the region is β. (These concepts are defined and optimum β-expectation tolerance regions and their power functions are developed for k-variate normal distributions in two of the given references.)

For the single exponential distribution with location parameter μ and scale parameter σ, similar β-expectation tolerance regions which are minimax and most stringent, and which have the ability to pick up 100 β per cent. sets on the right-hand tail of the distribution, are obtained for three cases:

(i) μ known ($= \mu_0$), σ unknown,
(ii) μ unknown, σ known ($= \sigma_0$), and
(iii) μ and σ unknown.

The β-expectation tolerance region for the double exponential distribution with μ known ($= \mu_0$) and which is constructed so that it has the ability to pick up the centre 100 β per cent. part of the distribution is also found. These four regions are $[\mu_0+\alpha_\beta\bar{x},\ \infty)$, $[x_{(1)}-b_\beta\sigma_0,\ \infty)$, $[x_{(1)}-c_\beta s,\ \infty)$ and $[\mu_0-d_\beta t,\ \mu_0+d_\beta t]$, respectively, where $\bar{x} = n^{-1}\sum_{i=1}^n x_i$, $x_{(1)} = \min_{i=1}^n x_i$, $s = (n-1)^{-1}\sum_{i=1}^n (x_i - x_{(1)})$ and $t = \sum_{i=1}^n |x_i - \mu_0|$ and where n is the sample size. The quantities α_β, b_β, c_β and d_β are tabulated for $n = 1(1)20$, 30, 40 and 60 and for $\beta = 0{\cdot}99$, $0{\cdot}95$, $0{\cdot}90$, and $0{\cdot}75$. The part of the distribution picked up by these tolerance regions is dependent on the chosen " measure of desirability ".

For the first three tolerance regions presented, the measure of desirability employed is a single exponential distribution with larger scale parameter, and for the last region, a double exponential distribution with smaller scale parameter is used. Expressions are presented for the power (degree of confidence) that the β-expectation tolerance regions cover the specified 100 β per cent. parts of the distributions (when the desirability of covering these parts is specified). Actual values of the power are tabulated for each of the four cases for a measure of desirability of $0{\cdot}995$ and for $n = 1$, 3, 5, 7, 10, 15, 30 and 60 and $\beta = 0{\cdot}99$, $0{\cdot}95$, $0{\cdot}90$ and $0{\cdot}75$.

The distributions that are necessary for the evaluation of α_β, b_β, c_β, and d_β are derived in an appendix.

(J. W. Wilkinson)

HOOPER, J. W. & THEIL, H. (Netherlands School of Economics, Rotterdam) **4.1** (6.1)

The extension of Wald's method of fitting straight lines to multiple regression—*In English*

Rev. Int. Statist. Inst. (1958) **26**, 37-47 (7 refs., 7 figs.)

In 1940 the late A. Wald [*Ann. Math. Statist.* (1940) **11**, 284-300] proposed a simple method for the estimation of a regression coefficient in the case of one explanatory variable. A first extension to multiple regression analysis was given by Gibson & Jowett [*Appl. Statist.* (1957) **6**].

The present authors propose an alternative extension to multiple regression analysis which can be described as follows. Just as in Wald's method, a straight line in a two-dimensional plane is determined by two centres of gravity of subsets of the material, so in the present method a plane in a three-dimensional space is determined by three centres of gravity. These are computed from three groups of observations such that no observation is contained in more than one group. The problem of how these groups should be chosen is solved in an optimal way, viz., such that the sampling variance of the estimator is a minimum. This is done under two alternative assumptions concerning the simultaneous parent distribution of the two explanatory variables. In both cases the pairs of observations on these variables are assumed to be uniformly distributed, viz., in the first case over an equilateral triangle, in the second case over a square.

The efficiency of this estimator is computed under both assumptions and compared with that of the Gibson-Jowett method, which appears to be superior in this respect. As to the computational effort, however, the present method is simpler.

(T. Kloek)

JÍLEK, M. & LÍKAR, O. (Central Res. Inst. of Food Ind. and Res. Inst. for Food Ind. Econ.) **4.5** (3.2)

Coefficients for the determination of one-sided tolerance limits of normal distribution—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11**, 45-48 (2 refs., 1 table)

A table of coefficients k is given so that the interval $(-\infty, \bar{x}+ks)$ or $(\bar{x}-ks, +\infty)$ covers with the probability p at least the ratio α of the normal population with unknown parameters (mean, μ and standard deviation, σ); where $\bar{x}$ is the sample mean and s is the unbiased estimate of the standard deviation. The authors use the fact that k has the form $k = (1/\sqrt{n})t(n-1, U_\alpha/n, 1-p)$, where $t(f, \delta, \epsilon)$ is the 100ϵ percentage point of the non-central t-distribution with f degrees of freedom and with the parameter of non-centrality δ. U_α is the $100(1-\alpha)$ percentage point of the normal distribution $N(0, 1)$. The values of coefficients k are tabulated for $n = 5(1)\ 20(5)\ 50(10)\ 100(100)\ 300$, 500 and 1000, for $p = 0{\cdot}90$, $0{\cdot}95$, $0{\cdot}99$ and for $\alpha = 0{\cdot}90$, $0{\cdot}95$, $0{\cdot}99$.

The method of calculation is essentially an iteration method on the basis of the tables of non-central t-distribution which were proposed by Johnson and Welch (1939) and Janko (1958). The first approximate of value t is calculated in accordance with the approximate formula for large sample. An auxiliary value λ, is determined from the non-central t-distribution tables. The second approximate value t_2 is calculated by substituting λ, in place of U_p of the asymptotic formula. The iteration is stopped when two successive approximations give the same value. The authors state that in about one-third of the cases agreement was reached in the second approximation.

(K. Isii)

KULLDORF, G. (Department of Statistics, University of Lund) **4.3** (2.8)

A problem of maximum likelihood estimation from a grouped sample—*In English*

Metrika (1959) **2**, 94-99 (4 refs.)

Two random variables X and Y have the distribution functions $F(x)$ and $G(x)$, and the relation $F(x) = \delta G(x)$ holds for all $x < x'$. The paper is concerned with the estimation of δ when $G(x)$ is known either completely or with the exception of p parameters. Suppose that n independent observations about X are made which are grouped into k specified intervals and let $x_0(= -\infty), x_1, \ldots, x_k(= \infty)$ denote the ordered group limits. It is assumed that $x_{k-1} = x'$, i.e. only in the last interval the relation does not hold.

When $G(x)$ is completely specified, it is a straightforward proof that the maximum-likelihood-estimate is $\hat{\delta} = (1-n_k/n)/G(x_{k-1})$, where n_k denotes the number of observations in the last interval.

The variance of maximum likelihood estimate $\hat{\delta}$ is also given and it is stated that n_k is a sufficient statistic for δ and complete, and that $\hat{\delta}$ is the uniformly minimum variance unbiased estimate of δ.

In the other case, where $G(x)$ is known—except the p parameters, it is shown that the maximum likelihood estimator of the parameter in this case is the same as the maximum likelihood estimator of the same parameters when a truncated sample of $n-n_k$ observations about the random variable Y is made and the truncation point is $x = x_{k-1}$. Here it is possible to use the known results about truncation for the estimation of the parameter. Then the maximum likelihood estimate of δ is the same as in the first case using in $G(x)$ the maximum likelihood estimate estimates for unknown parameters.

This estimation problem has occurred in a study of human migration [Kulldorf, *Migration Probabilities* (1955) Lund] where $G(x)$ was represented by the exponential, the Pareto and the logarithmic-normal distributions.

(E. Walter)

LEVERT, C. (Royal Meteor. Inst., de Bilt, Netherlands) **4.5** (11.3)

A nomogram for confidence intervals and exceedance probabilities—*In Dutch*
Statist. Neerlandica (1959) **13**, 3-14 (5 refs., 1 table, 3 figs.)

This paper deals with two problems regarding the probability β, that an observation of a normally (μ, σ^2)-distributed random variable exceeds a given value W. If the true mean value μ and the true variance σ^2 are not known (whereas the sample furnishes estimate $\bar{x}$ and s^2), these problems are as follows:

(i) If W is given, to determine a confidence interval for β;
(ii) If β is given, to determine a confidence interval for W, both intervals being based on a significance level α.

The problems are solved by means of an approximation for the exceedence probability of a random variable which follows the so called non-central t-distribution. This approximation enables us to construct a nomogram.

After having treated the problems theoretically, the author explains the construction of the nomogram. Next the application is illustrated by some numerical examples from the field of climatology; for example, with regard to the probability of a monthly quantity of precipitation surpassing the maximum ever measured and the reliability of such a probability.

In the final section the accuracy of the nomogram is studied by means of a comparison of figures read from the nomogram with the results furnished by the approximate formulas or given by the tables of the non-central t-distribution, published by Resnikoff and Liebermann [(1947) Stanford University Press]. The nomogram for three levels: $\alpha = 0{\cdot}005$, $0{\cdot}025$, and $0{\cdot}05$ are given. The curves are drawn for sample sizes of $N = 6$, 8, 10, 20, 30, 40, 50, 100, 500, 1000, 10,000 and ∞. The size of the nomogram is 17 cm. by 27 cm.

(C. Levert)

MALLOWS, C. L. (University College, London) **4.1** (4.7)

The information in an experiment—*In English*
J. R. Statist. Soc. B (1959) **21**, 67-72 (8 refs.)

Consider an experimenter A and experiments designed to increase A's knowledge about a parameter θ, an experiment E being taken to mean the observation of a random variable x whose distribution is assumed known (to A) except for the value of θ. It is attempted to construct a numerical measure of the information obtained from E, with the object of comparing different experiments.

It is assumed that (1) A's beliefs at any stage can be completely described by the specification of a distribution of θ (his *belief* distribution $b(\theta)$); (2) A proceeds by Bayes' theorem; (3) A's *knowledge* is an (unknown) function of his belief distribution and the true value of θ; (4) a numerical measure W of A's knowledge exists, this is called A's *information*. It follows that the average expected increase I in A's information can be regarded as the decrement in a risk function, and is additive for successive experiments.

It is further assumed that (5) the change in information is invariant under transformations of θ, and (6) the average change is non-negative. It is thought that there is essentially only one function W satisfying these assumptions, namely $W(b(.), \theta) = \log b(\theta)$ (a footnote is incorrect). The resulting quantity I can be identified with Good's " average weight of evidence " for the true value of θ, and is closely related to the measures proposed by Fisher, Kullback and Leibler, and Lindley.

A comparison of experiments is defined; it is shown that " E_1 is *sufficient* for E_2 " (in Blackwell's sense) implies " E_1 is *uniformly more informative* than E_2 " (in the present sense), and that this in turn implies " E_1 is *more informative* than E_2 " (in Lindley's sense); but neither implication can be reversed in general.

A sufficient statistic is " uniformly fully informative "; for each θ and $b(\theta)$ the likelihood ratio statistic is " locally fully informative ". Asymptotically unbiased, the normal estimators are ordered according to their variances. It is shown that the stopping rules for sequential binomial sampling which are obtained by keeping the maximum increase in information constant, are very similar to Lindley's.

The author distinguishes between the experimental situation and that arising in communication theory; it is claimed that here it is *not* appropriate to average over all possible values of θ, and that the quantity I is a satisfactory measure of " value of information ".

(C. L. Mallows)

MIHOC, G. (Bucharest University) **4.3** (8.1)

A method for determining the sample precision—*In Rumanian*
Revista de Statistică (1959) **5**, 11-23

This paper is the sequel to an earlier one entitled "The sampling method" [*Revista de Statistică* (1957), **12**, 21-35] dealing with the precision of the results in a sample, by making use of the Boole formula.

The paper is divided into two parts. In the first one a population A with two characteristics (a, b) is investigated using simple samples. The estimation of the precision of a/b is determined by means of the (estimated) precisions of the characteristics a and b. The method developed in this paper consists of determining the approximation relations (by excess and by loss) of a/b with the aid of the sample means and of the triple of the sample variances corresponding to a and b. Generally, this method yields a lower precision than that obtained by classical methods; nevertheless, this method is very simple and does not need so many calculations. When the characteristics a and b have a correlation coefficient near to -1, the method given here yields a precision near to that obtained by classical methods. Furthermore, by making use of the same method the sample precision of $a-2b/a+2b$ is determined.

In the second part of this paper the sample precision in stratified samples is estimated. The paper contains many interesting and instructive numerical examples illustrating this method in comparison with the classical ones.

(G. Sîmboan)

405

STEIN, C. (Stanford University, California) **4.4** (4.2)

An example of wide discrepancy between fiducial and confidence intervals—*In English*
Ann. Math. Statist. (1959) **30**, 877-880 (8 refs.)

This paper considers the following situation: "Let X_1, X_2, ..., X_n be independently normally distributed real random variables with unknown means ξ_1, ξ_2, ..., ξ_n and unit variance. We wish to construct fiducial or confidence sets for $\Sigma\,\xi_1^2$." These sets are constructed and it is shown that, for a large range of values of $\Sigma\,\xi_1^2$, a fiducial interval for this parameter having a given fiducial probability has the property that it is practically certain the interval will not cover the true parameter value. The sense of this is that as $n\to\infty$ with $1/n^2\,\Sigma\,\xi_1^2\to 0$, the probability that $\Sigma\,\xi_1^2$ is covered by the fiducial interval tends to zero.

The author concludes with several questions and comments which are aimed at the philosophy of fiducial probability statements. He takes the position that it is relevant to consider the probability of an interval estimator covering the true parameter value, even though this position is opposed by many practitioners of fiducial inference.

(D. H. Shaffer)

406

TRYBULA, S. (Math. Inst., Polish Academy of Sciences, Wroclaw) **4.0** (1.3)

The estimation of frequency in a population of elements belonging to classes not represented in the sample—*In Polish*

Zastosowania Mat. (1959) **4**, 244-248 (1 ref.)

Consider a population divided into a finite or denumerable number of classes. Let p_i be the frequency of the ith class in the population. Denote by $\mu_{r,n}$ and $\nu_{r,n}$ the total frequency of classes represented r times in a sample of size n taken according to a Bernoulli scheme and the number of such classes, respectively. It is proved that the random variable $\mu_{r,n}$ and the random variable

$$\Omega_{r,n}^{(m)} = \binom{n}{r} \sum_{k=0}^{m} \binom{m}{k} \nu_{k+r+1,n+m+1} \left(\binom{m+n+1}{k+r+1} \right), \quad (m = 0, 1, 2, \ldots),$$

have the same expected value. Owing to this it is proposed to use the variables $\Omega_{0,n}^{(m)}$ as estimators of the total frequency of classes not represented in a sample.

(S. Zubrzycki)

WIJSMAN, R. A. (University of Illinois, Urbana, U.S.A.) **4.2** (4.3)

On the theory of best asymptotically normal estimates—*In English*

Ann. Math. Statist. (1959) **30**, 185-191 (8 refs.)

The author uses a definition of best asymptotically normal estimates which requires less stringent assumptions, while retaining all the important theorems in the usual theory of best asymptotically normal estimates.

Let $\mathbf{Z}_n$ be a sequence of k-dimensional random vectors. The distribution of the $\mathbf{Z}_n$ depends on an m-dimensional parameter vector $\boldsymbol{\theta}$, $m \leqq k$. The true value of $\boldsymbol{\theta}$ is $\boldsymbol{\theta}_0$. It is assumed that the law $\sqrt{\mathbf{Z}n}[_n - \zeta(\boldsymbol{\theta}_0)]$ converges to the normal multivariate law, $N[0, \Sigma(\boldsymbol{\theta}_0)]$ in which $\Sigma(\boldsymbol{\theta}_0)$ is a k by k positive semi-definite matrix. $\boldsymbol{\theta}$ is estimated by $\hat{\theta}(\mathbf{Z}_n) = \hat{\theta}_n$.

The usual best asymptotically normal estimation theory assumes that both ζ and Σ have continuous second derivatives whereas the author's definition merely assumes Σ is continuous and ζ has a continuous first derivative, V; the other assumptions are unchanged, viz. ζ is one-to-one and bicontinuous, $\Sigma(\boldsymbol{\theta})$ is non-singular for every $\boldsymbol{\theta}$ and the matrix $\partial\zeta/\partial\boldsymbol{\theta}$ is of rank m for every $\boldsymbol{\theta}$. $\hat{\theta}$ is called *regular* (*1*) under the usual assumptions if

(i) $\hat{\theta}$ is a consistent estimator of $\boldsymbol{\theta}_0$;

(ii) $\partial\hat{\theta}/\partial z$ exists and is continuous in the neighbourhood of ζ.

A *regular* (*1*) estimate with minimal covariance matrix is called *best asymptotically normal* (*1*). The author points out that in order to generate *best asymptotically normal* (*1*) estimates by minimising a quadratic form, or from the root of a linear equation, it is practically necessary to require that Σ is continuously differentiable and ζ is continuously twice differentiable.

$\hat{\theta}$ will be called *regular* (*2*) under the new assumptions if (i) $\hat{\theta}$ is continuous in each point of ζ; (ii) for each θ, there exists an $m \times k$ matrix $\mathbf{A}(\boldsymbol{\theta})$, continuous in θ, such that $\sqrt{n}[\hat{\theta}_n - \theta_0]$ converges in probability to

$$\mathbf{A}(\boldsymbol{\theta}_0)\sqrt{n}[\mathbf{Z}_n - \zeta(\boldsymbol{\theta}_0)],$$

for all θ_0. A *regular* (*2*) estimate with minimum covariance matrix is called *best asymptotically normal* (*2*). The class of *regular* (*2*) estimates contains both the *regular* (*1*) estimates and the estimates defined by Ferguson [*Ann. Math. Statist.* (1958) **29**, 1046-1062]. Furthermore, the asymptotic properties of *best asymptotically normal* (*2*) estimates are the same as those of *best asymptotically normal* (*1*) whenever the latter exist. Thus the class of estimates considered is enlarged and if *best asymptotically normal* (*1*) estimate exists, it still belongs to the best estimate of this class. It is proved that $\mathbf{A}(\boldsymbol{\theta})\mathbf{V}(\boldsymbol{\theta}) = \mathbf{I}_{pm}$ identically in $\boldsymbol{\theta}$. This theorem ensures the existence of the minimal covariance matrix and shows that the class of covariance matrices of *regular* (*1*) and *regular* (*2*) estimates coincide. The author states that with a few modifications the whole theory of *best asymptotically normal* (*1*) estimates holds for *best asymptotically normal* (*2*) estimates.

(M. V. Menon)

BODMER, W. F. (University of Cambridge) **5.5** (6.9)

A significantly extreme deviate in data with a non-significant heterogeneity chi-square—*In English*
Biometrics (1959) **15**, 538-542 (10 refs., 1 table)

The chi-square test for heterogeneity may have little power for a particular alternative. An approximate test is considered for the alternative that one of the expected cell frequencies exceeds (or is less than) all the others, which are assumed equal.

For a set of k binomial frequencies p_i, each of the frequencies is transformed into an approximate normal variate. Then Nair's test for outliers is recommended to perform the desired test on the expected cell frequencies. A biological example is given in which the over-all chi-square is not significant, but the extreme observed cell frequency is significant using this method.

(C. P. Quesenberry)

CHANDA, K. C. (Bombay University) **5.0** (–.–)

On some simplification in the construction of similar regions—*In English*
Bull. Calcutta Statist. Ass. (1959) **8**, 159-161 (1 ref.)

If ϕ denotes the likelihood based on n observations of $p+k$ parameters $\theta_1, \theta_2, \ldots, \theta_{p+k}$, then to test the composite null hypothesis $H_0 : \theta_i = \theta_i^0$, $p+1 \leqq i \leqq p+k$, against the composite alternative hypothesis $H : \theta_i \neq \theta_i^0$, $p+1 \leqq i \leqq p+k$, it is known that under the conditions given by Neyman [*Ann. Math. Statist.* (1941) **12**, 46] on

$$\phi_0 = (\phi \mid H_0), \quad \psi_i = \frac{\delta \log \phi}{\delta \theta_i}\bigg|_{\phi=\phi_0} \quad (i = 1, 2, \ldots, p)$$

and

$$\psi_{ij} = \frac{\delta^2 \log \phi}{\delta \theta_i \delta \theta_j}\bigg|_{\phi=\phi_0} \quad (i, j = 1, 2, \ldots, p)$$

the complete class of similar regions w of size α can be constructed. The most powerful similar region is such that

$$\text{inside } w_0(\psi) : \phi \geqq \lambda \phi_0$$

where $w_0(\psi)$ is the part of w_0 inside the hypersurface $\psi_i = \text{const.}$ $(i = 1, 2, \ldots, p)$ and λ is suitably chosen.

By setting $\psi_{p+1} = \log \phi - \log \phi_0$, and suitably transforming the variables $\psi_1, \ldots, \psi_{p+1}$, the author shows that under some conditions the most powerful similar region w_0 is such that

$$\text{inside } w_0 : \lambda_{p+1} \geqq c \text{ (or } \leqq c')$$

where $\lambda_1, \ldots, \lambda_{p+1}$ are the new variables and c or c' is chosen suitably.

The method simplifies considerably the construction of similar regions. It is illustrated by applying the technique for testing the equality of means of k normal populations with equal variances against the most general alternatives. The author believes that the technique can be applied to more complicated cases with a considerable increase in ease.

(T. V. Hanumantha Rao)

GEISSER, S. (Nat. Inst. of Mental Health, Bethesda, Maryland) **5.8** (7.9)

A method for testing treatment effects in the presence of learning—*In English*
Biometrics (1959) **15**, 389-395 (6 refs.)

The usual test for equality of effects of p different drugs is a univariate analysis of variance. A sufficient number of experimental units (subjects) each receives one of the p drugs as a treatment and the analysis is performed on the resulting data. The author presents a multivariate alternative to this test when the experimental units are scarce.

The procedure is to apply all treatments to each experimental unit in order to obtain a reasonable number of observations. The dependence between observations on the same subject violates the assumptions of the univariate analysis of variance and requires a multivariate approach. The observations on each individual are treated as a single vector observation.

To prevent confounding of the treatment (drug) effects with the time (order of application of drugs) effects a latin square design is used. The multivariate T^2 statistic is used to test the hypothesis that all drugs have the same effect.

The assumptions for this test are that each observation is a linear combination of the appropriate treatment effect, a time (learning) effect due to the order in which the drug is applied and an error term. It is further assumed that the vector of observations for the jth subject is distributed with mean vector μ_j and variance-covariance matrix identical for each subject.

The author suggests a Youden square arrangement to handle the problem of missing data and considers two possible alternative models, a multiplicative model and a partially multiplicative, partially additive model. The author also states that the variance-covariance matrices V_j of the p groups are identical ($V_j = V$) under the assumption of a linear model and unequal ($V_j \neq V$) if the true model is one of the above alternatives. Hence, in order to test the additivity of the model versus these alternatives an equivalent test of the homogeneity of the variance-covariance matrices V_j may be used. The author refers the reader to such a test and suggests that the T^2 statistic might still be used if one of the alternative models is accepted. This leads to an approximate χ^2 test.

(H. O. Posten)

411

ISHII, G. (Dept. of Statistics, Atomic Bomb Commission, Hiroshima) **5.6** (3.8)

On the exact probabilities of Rényi's tests—*In English*
Ann. Inst. Statist. Math., Tokyo (1959) **11**, 17-24 (9 refs.)

In this paper two exact probabilities are obtained by the author,

(i) $\Pr\{\sup_{b \geqq F(x)} [F(x)-F_n(x)/1-F(n)] < \epsilon\}$ and

(ii) $\Pr\{\sup_{b \geqq F(x)} [F_n(x)-F(x)/1-F(x)] < \epsilon\}$, where $F(x)$ is a continuous type distribution function and $F_n(x)$ is an empirical distribution function based on a sample of size n from the distribution $F(x)$.

Formerly Rényi (1953) treated the statistics

$$\sup_{a \leqq F(x)} [F_n(x)-F(x)/F(n)] \text{ and } \sup_{1-a \geqq F(x)} [F(x)-F_n(x)/1-F(x)],$$

obtained their limiting distributions when n tends to infinity, and showed that these limiting distributions are distribution-free. The treatment by the present author differs from that of Rényi in the method employed: in particular, he considers the case where n is finite. The results show that, as in Rényi's case, the exact probabilities are also distribution-free; that is, independent of the form of $F(x)$.

The author states that these results will be useful for a test of goodness-of-fit in life testing. [See also: Ishii "Test of fit in life testing" *Ann. Inst. Statist. Math., Tokyo* (1957) **9**, 117-125 and "Kolmogoroff-Smirnoff test in life test" *Ann. Inst. Statist. Math., Tokyo* (1958) **10**, 37-45].

The method of his proof can be summarised as follows. Let the ordered sample of values from $F(x)$ be $x_1, x_2 \ldots x_n$, and X a random variable following $F(x)$. Transform x_i by $F(x)$, and put $U_i = F(x_i)$. Then, U_i is a realised value of the random variable $U_i = F(X)$ which follows the uniform distribution over the interval [0, 1]. By this transformation, the relation $\sup_{b \geqq F(x)} [F(X)-F_n(X)/1-F(X)] < \epsilon$ is changed into $\sup_{b \geqq U} [1-G_n(U)/1-U] < \epsilon$, where $G_n(U_i) = (i-1)/n$.

The latter relation is equivalent to the following (R) on $U_1, U_2, \ldots, U_n$ under the condition that $U_m < b < U_{m-1}$.

$$(R) \begin{cases} U_i < [(i-1)/n+\epsilon]/1+\epsilon & \text{if } i \leqq k+1 \\ U_i < b & \text{if } k+1 < i \leqq m. \end{cases}$$

where k is an integer such that $\{k/n+\epsilon\}/1+\epsilon \leqq b < \{(k+1)/n+\epsilon\}/1+\epsilon$. For a fixed number $b(0 < b \leqq 1)$, a suffix m with $U_m < b < U_{m+1}$ is also a random variable which follows the binomial distribution $B(b, n)$. The joint distribution of $U_1, \ldots, U_m$, and m is $n!/(n-m)!(1-b)^{n-m} du_1, du_2, \ldots, du_m$. By integrating this under the above mentioned restriction (R) and summing them up for all m with $n \geqq m \geqq k$, the author obtains his results. Concerning the above integration, several lemmas are stated and the main lemma is proved.

The exact value of the other probability (ii) is obtained analogously.

(Y. Suzuki)

412

KAPUR, M. N. (Indian Council of Agric. Res., New Delhi) **5.4** (-.-)

A property of the optimum solution suggested by Paulson for the k-sample slippage problem for the normal distribution—*In English*

J. Indian Soc. Agric. Statist. (1957) **9**, 179-190 (1 ref.)

In 1952, Paulson suggested [*Ann. Math. Statist.* **23**, 610-616] a test criterion for testing the hypothesis that k given normal populations are identical against alternatives under which the mean of one is greater than the mean of all the others. If $x_1, x_2, \ldots, x_k$ are samples of size one from each of the k populations, and if S^2 is an independent estimate with n degrees of freedom of their common variance, the procedure suggested is to take the decision D_i that the ith population has the greatest mean if

$$(x_M - \bar{x})\left\{knS^2 + (n+1)\sum_{i=1}^{k}(k_i - \bar{k})^2\right\}^{-\frac{1}{2}} \geqq c \quad (i = 1, \ldots, k)$$

and to take the decision D_0 that all populations have the same mean when

$$(x_M - \bar{x})\left\{-knS^2 + (n+1)\sum_{i=1}^{k}(x_i - \bar{x})^2\right\}^{-\frac{1}{2}} < c$$

where $x_M = \mathrm{Max}(x_1, \ldots, x_k)$, $\bar{x} = (x_1 + \ldots + x_k)/k$.

Paulson showed that this procedure was unbiased. In this paper the author proves that the test procedure is unbiased even when the class of alternatives is the wider class of k normal populations with equal variances but which contains at least one pair with unequal means. To be more precise, the author proves the following:—

(i) $P[D_0 \mid w_0] \geqq P[D_0 \mid w]$
(ii) $P[D_i \mid w_i] \geqq P[D_j \mid w_i]$ $i, j = 1, 2, \ldots, k$ $(i \neq j)$
(iii) $P[D_i \mid w_i] \geqq P[D_i \mid w_0]$ $(i = 1, 2, \ldots, k)$.

where $w_i (i = 1, 2, \ldots, k)$ is that portion of the parameter space where the ith population has the greatest mean.

(S. John)

KONIJN, H. S. (University of Sydney, Australia) **5.6** (5.3)

Some nonparametric tests for treatment differences in paired replications—*In English*

J. Indian Soc. Agric. Statist. (1957), **9**, 145-167 (50 refs.)

Consider n pairs of random variables $(W_i, V_i)(i = 1, 2, \ldots, n)$ and the hypothesis H_0 that for each i the distribution of $Z_i = W_i - V_i$ is symmetrical about zero against the alternative that it is symmetrical about some $\theta_i \neq 0$. For example W_i and V_i might be the responses of the treated and untreated members in the ith pair of experimental units and H_0 the hypothesis of "no treatment effect". The alternative envisages a situation where the treatment effects the ith pair only in so far as it shifts the response of its treated member by θ_i.

Let z_i be the observed values of Z_i. Under H_0, the number m of positive z's obeys a binomial distribution with parameters $\frac{1}{2}$ and n and the division of the observed differences into those with positive and negative signs constitutes a random partitioning of $|z|$'s. It has therefore been pointed out that any two-sample test which is symmetric in the observations, can be used to construct tests for the symmetry hypothesis. The Fisher-Pitman randomisation analogue of the paired t test can be (trivially) seen to be a test of this type, and in fact Wilcoxon's matched pair signed rank test is derived thus from the two-sample rank sum test. One might similarly extend the Median test or the Mann-Whitney test.

In this paper Konijn studies the Pitman asymptotic power of all these tests and also the sign test which has no counterpart in the two sample theory. In deriving his expressions for the local asymptotic power, Konijn assumes that the shift θ_i is the same (numerically) in all the pairs and that the different pairs of random variables are independently distributed. The following comments are made:

(i) *Sign test.* In general the efficiency of the sign test is highly dependent on the local properties of the density of the distribution and has a greatest lower bound of zero, so that the sign test is not in general to be recommended as a symmetry test against shift alternatives. In the normal case (that is when the Z_i's are normally distributed with the same variance) the sign test has a Pitman local asymptotic efficiency of $2/\pi = 0{\cdot}637$ (approx.) with respect to the randomisation test. If the different pairs are identically distributed and observations within pairs are independent, the local asymptotic efficiency has a minimum of 0·576.

(ii) *Rank sum and Mann Whitney test.* In the normal case, the local asymptotic efficiency with respect to the randomisation test are $3/\pi = 0{\cdot}995$ (approx.) for the rank sum test and 0·32 for the Mann-Whitney test.

(iii) *Median test.* In the normal case it has a local asymptotic efficiency of 0·81 with respect to the randomisation test. In general, however, its efficiency, like that of the sign test, has the undesirable property of depending strongly on the local properties of the density.

(S. K. Mitra)

KONIJN, H. S. (University of Sydney, Australia) **5.2** (6.0)

Positive and negative dependence of two random variables—*In English*
Sankhya (1959) **21**, 269-280 (5 refs.)

Given the marginal distributions G and H of two random variables X, Y, Frechet [*Ann. de l'Univ. Lyon* (1951), 53-77] proved the existence of a joint distribution function $F_+(F_-)$ which is the uniformly highest (lowest) valued among all possible joint distribution functions of X, Y and which implies an almost sure non-decreasing (non-increasing) relation between X and Y. The random variables are defined to be completely positive (negative) dependent if there is an almost sure non-decreasing (non-increasing) relation between the two and positively k-dependent if their joint distribution can be expressed as $(1-k)F_0+kF_+$, where $F_0 = GH$ is the joint distribution of the variables if they are independent. Negative k-dependence is similarly defined. This paper discusses the properties of such variables and the properties of various tests for independence against such types of dependence.

The test statistics considered in this paper are the estimates of

(i) Correlation coefficient ρ,
(ii) Spearman's rank correlation τ_0,
(iii) Difference sign correlation τ_1,
(iv) Grade correlation τ_2,
(v) Medial correlation τ_3.

The author derives classes within which the first four statistics yield consistent tests of independence.

ρ, τ_1, τ_2 and τ_3 are computed for positively and negatively k-dependent random variables. Further, it is shown that the tests based on unbiased estimates of the τ are unbiased. Powers of these tests against alternatives in k are obtained and also their asymptotic power when k is near zero. The latter turns out to be independent of the marginals.

(K. Rajaraman)

415

KUIPER, N. H. (Landbouwhogeschool, Wageningen) **5.6** (1.4)

Tests concerning random points on a circle—*In English*
Proc. Kon. Ned. Akad., Wetensch. A (1960) **63**.
Indag. Math. (1960) **22**, 38-47 (8 refs., 3 tables, 1 fig.)

Let $M = (\phi_1, \ldots, \phi_m)$ be a sequence of angles $(0 \leqq \phi_i < 2\pi)$ representing compass-directions into which m birds have been seen flying on migration at a given place and time. In this paper tests concerning the null-hypothesis that $\phi_1, \ldots, \phi_m$ are independent values of a known random angle ϕ (problem 1), and tests for the hypothesis that two sets M and M^1 belong to the same unknown random angle (problem 2) are studied.

In the first part of this paper three obvious methods are considered:

(i) In the case where the probability mass is concentrated in a small ϕ-interval, the circle of angles (mod 2π) can be cut, and methods for ordinary random variables applied (problem 2).
(ii) The interval $0 < \phi \leqq 2\pi$ can be divided in subintervals and the chi-squared goodness-of-fit test can be applied (problem 1).
(iii) The random vector $z = (\cos\phi, \sin\phi)$ can be considered as a two dimensional random vector (problem 1 and 2). If ϕ has a uniform distribution then the statistic $\sum_{i=1}^{m} z_i$ is asymptotically "isomorous" to $\chi_2\sqrt{(\frac{1}{2}m)}$ where "isomorous" stands for "having the same cumulative distribution function" and chi-square is the two-dimensional normal random vector with expectation zero and unit covariance matrix. Estimation problems are also considered here.

In the second part of the paper, non-parametric tests on the circle related to the tests of Kolmogoroff and Smirnoff are introduced and studied. A statistic V_n and a statistic $V_{n,m}$ are suggested and tables of critical regions are given for the V_n-test (problem 1) and the $V_{n,n}$-test (problem 2).

(N. Kuiper)

SIOTANI, M. (Inst. Statist. Math., Tokyo) **5.5** (5.9)

The extreme value of the generalised distances of the individual points in the multivariate normal sample—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **10**, 183-208 (12 refs., 3 tables, 2 figs.)

This paper is concerned with the extreme value of the generalised distances from the origin of N individual points which may be correlated with each other in a p-variate normal sample. Let $Y_\alpha(\alpha = 1, 2, \ldots, N)$ represent N p-variate vectors with mean zero and with covariance matrix $Y\mathbf{\Lambda}(Y>0)$, where $\mathbf{\Lambda}$ is a symmetric, positive definite matrix. Further, let the covariance matrix of Y_α and $Y_\beta(\alpha \neq \beta)$ be $\delta\mathbf{\Lambda}$, where $|\delta|<Y$, and define the quadratic forms $Y_\alpha^1 A^{-1} Y_\alpha = Q_\alpha$ and $Y_\alpha^1 L^{-1} Y_\alpha = \hat{Q}_\alpha$, where L is an unbiased estimator of $\mathbf{\Lambda}$.

The statistics treated in this paper are $Q_{\max} = \max_\alpha (Q_\alpha)$ and $\hat{Q}_{\max} = \max_\alpha (\hat{Q}_\alpha)$. The extreme deviate from the population mean or from the sample mean in the sample of size N drawn from the multivariate distribution, the extreme deviate from the control variate and the range in the multivariate case are considered as special cases of $Q_{\max}$ and $\hat{Q}_{\max}$.

The main problem discussed in this paper is the obtaining of asymptotic formulas by which good approximate upper percentage points of $Q_{\max}$ and $\hat{Q}_{\max}$ can be calculated when Y has the p-variate normal distribution. For this, $\Pr[Q_\alpha > a^2, Q_\beta > a^2]$ and $\Pr[\hat{Q}_\alpha > B_1^2, \hat{Q}_\beta > B_1^2]$, where $\alpha \neq \beta$, are evaluated. The former is calculated from the bivariate chi-square distribution and the latter is asymptotically evaluated in terms of ν^{-1}, where ν is the number of degrees of freedom of L, by applying a slightly modified method by James [*Biometrika* (1954) **41**, 19-43]. The extreme deviate of N individual points from the sample mean is given in some detail. Tables of the upper 5, 2·5 and 1 percentage points are given for $p = 2, 3, 4$; $N = 3(1)10(2)20, 25, 30$ when $\mathbf{\Lambda}$ is known and for $p = 2$; $N = 3(1)12, 14$; $\nu = 20(2)40(5)60, 100, 150, 200$ when $\mathbf{\Lambda}$ is unknown. Finally, the author comments on the evaluation of upper percentage points of the square of the range.

(Y. Suzuki)

STERLING, T. D. (University of Cincinnati) **5.0** (11.0)

Publication decisions and their possible effects on inferences drawn from tests of significance—or vice versa—*In English*

J. Amer. Statist. Ass. (1959) **54**, 30-34 (8 refs., 2 tables)

In this paper the author discusses three theses:

(i) Experimental results will be printed with a greater probability if the relevant test of significance rejects H_0 for the major hypothesis with $\Pr(E \mid H_0) < 0{\cdot}05$ than if they fail to reject H_0 at that level.

(ii) The probability that an experimental design will be replicated becomes very small once such an experiment appears in print.

(iii) A great many more experiments are performed than appear in the pages of professional journals.

These theses are partially supported by a survey of 362 recent research reports in psychology which revealed that of the 294 that used tests of significance, only 8 failed to reject some "major" hypothesis (complicated conventions attach to the word "major"), and none were reports of replications of previously published experiments.

The preference for publication of significant rather than insignificant results may be either official journal policy or merely an implicit agreement among authors. In either case research leading to non-significant results may be repeated by various workers, each unaware of earlier negative results, until by chance a significant result occurs, leading to publication of an "error of the first kind."

The malpractices are not the private domain of psychology but are widespread also in experimental journals in biology, chemistry, medicine, physiology, and sociology. Some onus appears to be attached to the reporting of negative results.

(R. J. Buehler)

STEYN, H. S. (University of South Africa, Pretoria) **5.2** (2.4)

On χ^2-tests for contingency tables of negative multinomial type—*In English*
Statist. Neerlandica (1959) **13**, 433-444.

Usually, when in a contingency table a χ^2-test is used to test independence between the classifications, the total size of the sample is constant. The author treats the case where the number of observations in one of the cells is fixed and consequently the total size of the sample is a random variable. This situation occurs, for instance, when one expects one cell to have a very small probability and decides to continue sampling until a given number of observations in this cell is reached. Instead of the multinomial distribution connected with the ordinary contingency table the distribution of cell frequencies is now negative multinomial [a generalisation of the one-dimensional Pascal distribution, see: Steyn, *Proc. Kon. Ned. Akad., Wetensch.* A (1951) **54** and (1955) **58**].

Let us in a h by k contingency table denote by p_{ij} the probability of an observation to fall in cell c_{ij}, by m the fixed number in cell c_{00} and by f_{ij} the number of observations in cell c_{ij}. According to the negative multinomial distribution the expected frequency in cell c_{ij} is mp_{ij}/p_{00} and the expected total number of observations m/p_{00}. The expression χ^2, formed by summing up $(f_{ij}-\mathscr{E}f_{ij})^2/\mathscr{E}f_{ij}$ over all cells and subtracting the corresponding term for the total number of observations, has (asymptotically for $m\to\infty$) a χ^2-distribution with $hk-1$ degrees of freedom as Steyn proved in 1955. If the classifications are independent: $p_{ij}=p_ip_j$. In the present paper two methods are given to derive estimates for the unknown parameters p_i and p_j; the first using the row and column totals and the second using the method of modified minimum χ^2 [see Cramér, *Mathematical Methods of Statistics* (1954)]. In both cases the resulting expression after substituting the estimates in χ^2 has asymptotically a χ^2-distribution with $(h-1)(k-1)$ degrees of freedom and thus can be used to test independence between classifications. The proof of this statement is indicated. The author especially treats the fourfold or 2 by 2 table.

(Gerda Klerk-Grobben)

UHLMANN, W. (Inst. für Versich.-math. und Math. Statist., Hamburg) **5.6** (5.3)

On a nonparametric test by Lehmann—*In German*
Metrika (1959) **2**, 169-185 (6 refs.)

In 1953 [*Ann. Math. Stat.* **24**, 23-43] Lehmann showed that the Wilcoxon (Mann-Whitney) test is the locally most powerful rank test for the hypothesis $F_1(x)=F_2(x)$ against the alternative $F_1(x)=(1-p)F(x)+pF(x)^2$, $F_2(x)=F(x)$. In the same paper, Lehmann suggested a two-sided rank test which is locally most powerful for the alternative "$F_1(x)=(1-p)F(x)+pF(x)^2$, $F_2(x)=F(x)$ or $F_1(x)=F(x)$, $F_2(x)=(1-p)F(x)+pF(x)^2$". The power is defined in terms of the "average" power function $\beta(p)=\frac{1}{2}(\beta_1(p)+\beta_2(p))$, where $\beta_1(p)$ and $\beta_2(p)$ are the power functions under the first and second alternative, respectively. As $\beta'(0)=0$, the locally most powerful test is characterised by $\beta''(0)=$ Max. The test obtained in this way is, in general, different from the two-sided Wilcoxon test.

This present paper contains a table for this test with critical regions for all levels up to 0·05 for the sample sizes 4(1)9. Several lemmas of importance in connection with the computation of this table are given. It is seen that, for sample sizes up to 9 and levels of significance $\leqq 0{\cdot}05$, to each critical region of the two-sided Wilcoxon-test there corresponds a critical region of Lehmann's test at the same level. But Lehmann's test has additive critical regions, so that a given level of significance could be approached more accurately which might be important in the application of non-randomised tests.

(J. Pfanzagl)

Remark of the Regional Editor:

The abstractor is of the opinion that there exists a concept of "optimality" (different from the one based on the "average" power function, as used by Lehmann and Uhlmann), for which the two-sided Wilcoxon-test turns out to be optimal for the problem stated above. A short note on this subject will appear in *Metrika* (1960) **3**.

(J. Pfanzagl)

VAS, É. (Math. Inst., Hungarian Academy of Sciences, Budapest) **5.7** (8.7)

On the efficiency of the sequential probability ratio test—*In English*

Publ. Math. Inst. Hung. Acad. Sci. (1959) **4**, 171-181 (6 refs., 3 tables)

The author improves some results of Wald [*Sequential Analysis*, (1948) New York, Wiley] concerning the average sample number of his sequential probability ratio test. In the relevant formulas of Wald the following quantities appear:

$$\xi_\theta = \sup_{r \geqq 0} M\{z - r \mid \theta;\ z \geqq r\},$$

$$\xi'_\theta = \inf_{r \geqq 0} M\{z + r \mid \theta;\ z \leqq -r\},$$

$$\zeta_\theta = \sup_{r \geqq 0} M\{(z-r)^2 \mid \theta;\ z \geqq r\},$$

$$\zeta'_\theta = \sup_{r \geqq 0} M\{(z+r)^2 \mid \theta;\ z \leqq -r\},$$

where $z = \log \{f(x, \theta_1)/f(x, \theta_0)$.

Here $f(x, \theta)$ denotes the probability density function of the observed random variate x, θ denotes the unknown parameter, θ_0 and θ_1 its values for the two hypotheses. Wald calculated ξ_θ and ξ'_θ for the normal and binomial distribution.

The author proves that if z has a logarithmically concave density function (in the interval of its possible values) then the above extreme values appear at $r = 0$. This condition is satisfied if the parent distribution belongs to several important distributions, among them being the case of the normal parent distribution. For this case the author's result simplifies the calculations and gives better results. Tables are given for the approximate values and the lower and upper limits of the efficiency (average sample number) of the sequential probability ratio test for the case where the parent distribution is the normal distribution.

(K. Sarkadi)

VINCZE, I. (Math. Inst., Hungarian Academy of Sciences, Budapest) **5.6** (5.3)

On some joint distributions and joint limiting distributions in the theory of order statistics, II—*In English*

Publ. Math. Inst. Hung. Acad. Sci. (1959) **4**, 29-47 (3 refs., 1 table)

Let $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ be samples from populations having the distribution functions $F(x)$ and $G(x)$. Let us denote further by $F_n(x)$ and $G_n(x)$ respectively the empirical distribution functions of these samples.

Let $\xi_0^{(n)}$ in the following denote those smallest values of x_1 for which the function $F_n(x+0) - G_n(x+0)$ takes its maximum.

Let us put further

$$D^+_{n,n} = \max_{-\infty < x < \infty} (F_n(x) - G_n(x))$$

$$R^+_{n,n} = \tfrac{1}{2}[F_n(\xi_0^{(n)}+0) + G_n(\xi_0^{(n)}+0)]$$

In a previous paper [*Publ. Math. Inst. Hung. Acad. Sci.* (1957) **2**, 183-209] the author derived the joint distribution of the pair of random variables $(D^+_{n,n}, R^+_{n,n})$ as determined under the assumption $F(x) \equiv G(x)$. This paper derives the generating function of the probabilities $P^{(n)}_{r,k} = P\{D^+_{n,n} = k/n,\ R^+_{n,n} = \lambda/2n\}$ and some of its consequences.

Using the notation $P^{(n)}_{0,0} = 0$, the author states the generating function which has the form of a triple summation in n, k and λ. The marginal distribution of the probabilities with regard to k gives the distribution of Gnedenko-Koroljuk. In section 3 a direct proof of the identities

$$\sum_{(r)}{}^{*} P^{(n)}_{r,k} = \frac{2k+1}{n+k+1} \frac{\binom{2n}{n-k}}{\binom{2n}{n}}$$

is given and the conditional expectation of the random variable $R^+_{n,n}$ under the condition $D^+_{n,n} = k/n$ is also determined. The author then introduces the following notation:

$$\sum_{(k)}{}^{*} P^{(n)}_{r,k} = P^{(n)}_{r_1.}$$

and in the fourth section the following relation is proved for $q = 1, 2, \ldots, n$

$$P^{(n)}_{2q-1.} = P^{(n)}_{2q.}$$

A table is given for the probabilities $P^{(n)}_{r,k}$ where $n = 10$.

(E. Csáki)

WALSH, J. E. (System Development Corporation, Santa Monica, California) **5.5** (5.6)
Large sample nonparametric rejection of outlying observations—*In English*
Ann. Inst. Statist. Math., Tokyo (1959) **10**, 223-232 (4 refs.)

The author presents a method of nonparametric rejection of outlying observations. When $x_1 \leqq \ldots \leqq x_n$ are independent observations from continuous populations, the null hypothesis is that these observations are a sample from one and the same population. The alternate hypothesis is that the i smallest observations are too small (or that i largest observations are too large) to be consistent with the null hypothesis. However the continuous populations considered here are slightly restricted to types in which after the paper of David & Johnson [*Biometrika* (1954) **41**, 228-240], the mean and variance-covariance expansions, and some related Taylor expansions are valid. The test criterion is that the alternative is accepted if a statistic of form $x_i-(1+A)x_{i+1}+Ax_k$ is negative (or if $x_{n+1-i}-(1+A)x_{n-i}+Ax_{n+1-k}$ is positive) where $A>0$, k is the largest integer which is less than $(i+\sqrt{2n})$ and n is sufficiently large.

Consider a statistic of the form

$$X = x_i-(1+A)x_j+Ax_k, \quad 1<j<k \text{ and } A>0$$

then the problem is to select j, k, and A for given values of i and n, so that a satisfactory nonparametric test is obtained. The relation $j = i+1$ is established by requiring that $\{1+O(1)\}\,\mathrm{Var}\,(X \mid H_0)$ be minimum. The values of k and A are determined by this requirement and the additional conditions that $A>0$ and under H_0

$$E(X) = k\sqrt{\mathrm{var}\,(X)}\,\{1+O(1)\},$$

where K is specified.

An approximate upper bound for its significance level when n is not too small, is obtained by the relation

$$P\{x_i-(1+A)x_{i+1}+Ax_k<0\} = P\{X-\mathscr{E}(x)/\sqrt{\mathrm{var}\,(X)}< -K+O(1)\} \leqq 1/K^2+O(1)$$

under the null hypothesis. In the derivations, two assumptions are used, the first of which is introduced for deriving the above inequality and the second for the sake of mathematical completeness.

A two-sided test is obtained as a combination of these one-sided tests, and some power function properties of them are considered for the case where the observations are independent and are from populations with fixed shapes.

(H. Hudimoto)

WILLIAMS, E. J. (North Carolina State College, Raleigh) **5.8** (6.1)
The comparison of regression variables—*In English*
J. R. Statist. Soc. B (1959) **21**, 396-399 (2 refs.)

The problem discussed is that for which Hotelling originally put forward a test of significance. When a regression relationship for predicting the values of a variable is to be determined, and two or more regression variables have been measured, it is usually thought appropriate to calculate the multiple regression in order to obtain the predicted values. Sometimes it so happens that a choice of variables to be used for the predictor has to be made, and Hotelling's test was concerned with comparing the efficiency of the different regression variables. The author discusses Hotelling's test, derives it anew and expresses it as a test of regressions. It is pointed out that a test proposed by Healey is different from that of Hotelling.

A sample of n values of each of p regression variables $x_1, x_2, \ldots, x_p$ is supposed, and y an independent variable: z_i is the square root of the sum of squares about the regression of y and x_i. The test criterion is the sum of squares of the deviations of the z_i from their weighted means. This is tested against the residual mean-square. The author gives an analysis of variance table for splitting up the various sums of squares.

The limitations of Hotelling's test are noted. It is a conditional test and is considered not suitable for drawing conclusions about the future behaviour of the independent variables. The author proposes an approximate test for comparing two predictors.

(Florence N. David)

AOYAMA, H. (Inst. Statist. Math., Tokyo) **6.4** (6.9)

On the evaluation of the risk index of the railroad crossing—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **10**, 163-180 (23 tables, 4 figs.)

The author investigates the traffic volume at railroad crossings which belong to the Japanese National Railways, and he studies the method of calculating the risk index of crossings.

There are four kinds of crossings classified according to their physical equipment. Crossings of 2nd kind are not included in this paper, because they are very rare. Firstly, twenty-three characteristic factors of the railroad crossings are considered. Later, by comparing the distribution of the crossings of the 4th kind without an accident with that of crossings of the same kind with an accident and using the chi-square test, 8 main factors are selected. Numerical values to be given to the categories of these factors are calculated by maximising each correlation ratio. After these operations, the risk index y of any crossing is given by the equation with unknown weight w_i,

$$y = w_1x_1+w_2x_2+\ldots+w_8x_8$$

where x_i is the numerical value of the category taken by the ith factor of the crossing. The weight (w_i) is calculated by the method of linear discriminant functions. The author defines the coefficient of accident by the percentage of the number of crossing with accidents to that of all crossings having the same value of risk index and he gives the relation between it and the risk index. The risk index or the coefficient of accident is used as a criterion to alter the kind of crossings.

(Y. Suzuki)

ARMITAGE, P. (London School of Hygiene) **6.8** (4.2)

Host variability in dilution experiments—*In English*

Biometrics (1959) **15**, 1-9 (11 refs.)

Following Moran [*J. Hyg., Camb.* (1954) **52**, 189-193 and 444-446] and Armitage & Spicer [*J. Hyg., Camb.* (1956) **54**, 401-414] a re-examination is undertaken of the theory of experiments in which various concentrations of infectious particles are administered to groups of host organisms and where the response variable is the proportion p of hosts infected at each dose. Interest centres about the density function $f(p)$: estimation of parameters of $f(p)$ and testing the null hypothesis H_0: p = constant.

Correction is made of a previous assertion that a maximum likelihood approach was inappropriate. It appears that second derivatives of the likelihood function (given H_0 true) do converge. Discussion of the maximum likelihood approach and of certain tests of H_0 are given.

For analytical convenience (although theoretically objectionable except as an approximation) it is assumed that, for a given dose of infectious particles, $f(p)$ has a Type III distribution. The likelihood equations and expectations of the second derivatives are found. On H_0 these expectations are also given. When the number of hosts are constant for each dose and when dilutions are two-fold over a wide range, the summations appearing in the expectations on H_0 are close to the corresponding integrals. These are evaluated giving very simple expressions involving a single parameter γ. The asymptotic test of H_0 involving maximum likelihood estimates of parameters V and γ and the expectations is mentioned. An asymptotically equivalent test on H_0 is described not requiring evaluation of the maximum likelihood estimates. In this test use is made of a consistent but inefficient estimate. Fisher's estimate for the "dilution method", although not strictly consistent, is proposed as an alternative.

The data of Parker treated earlier is analysed: H_0 is strongly rejected and hence asymptotic equivalence of the tests is not attained. An example, artificially obtained, with H_0 nearly true is examined. The maximum likelihood test described above is found to be more efficient than two quick tests earlier proposed by Armitage & Spicer. This is due to the flattening of the response curve occurring even with small variability in p.

With regard to the choice of estimation versus testing H_0, two situations are considered: (i) the particles are of two kinds, infective and non-infective; (ii) some of the particles at least may be either infective or inert. In situation (i) it is reasonable that H_0 should be true; however for situation (ii) H_0 is inherently implausible and hence would never be tested. When the situations cannot be distinguished on *a priori* grounds a test of H_0 is tantamount to a test of situation (i). Slight departures from situation (i) may require modification of $f(p)$. The truncated exponential distribution is considered and is found to lead to the same test for H_0. It is suggested that the proposed test has validity unrestricted by strong assumptions about the form of $f(p)$.

(M. E. Turner)

ASHFORD, J. R. (National Coal Board, London) **6.8** (4.3)

An approach to the analysis of data for semi-quantal responses in biological assay—*In English*

Biometrics (1959) **15**, 573-581 (8 refs., 1 table, 2 figs.)

In connection with the problem of relating an observed quantal response to an underlying quantitative reaction Hewlett & Plackett [*Biometrics* (1956) **12**, 72-78] have shown that the quantal dosage-response relationship may be derived from the corresponding quantitative dosage-response relationship.

The purpose of this paper is to examine the application in reverse of some of the results given by Hewlett & Plackett in order to determine the effect of errors of measurement of both dosage and reaction on the quantal relationship. The paper also examines the problem of estimating the parameters of the underlying quantitative dosage-reponse relationship from the data of a multinomial quantal response.

The author discusses briefly the corresponding quantitative and quantal relationships. The effects of errors of measurement are discussed in some detail, and it is pointed out that observations of the response are likely to be in error, resulting in an increase in the variability of the quantitative dosage-response relationship. In some circumstances the dosage metameter may be subject to errors of measurement: this will in turn modify the relationship between the observed dosage metameter and the observed response metameter. The means and variances of the dosage and response metameters under the condition of linear relationship are given. Finally, under the above conditions, the author states that the " apparent " dosage response relationship is also linear although the slope of the regression line is reduced.

The quantal responses most frequently encountered in practice are of the binomial type, but there are some applications in which the underlying quantitative response scale may be subdivided into a number of ordered and mutually exclusive intervals and it is necessary to consider a multinomial response. At this point the author demonstrates expressions and calculations using maximum likelihood estimates and shows that the parameters associated with the underlying quantitative response may be estimated from the semi-quantal response. This statement is valid only if it is known which parameters are to be estimated: the method of analysis depends on the validity of the large number of assumptions which it is necessary to make.

The practical example used to illustrate the author's procedure was provided by an investigation of the causes of pneumoconiosis among coal miners from collieries in England. An iterative procedure was employed to solve the maximum likelihood equations and estimates of the parameters were obtained.

From the figure showing parallel quantal response lines the author concludes that better estimates of these lines may be derived from this method of analysis than would be obtained by a consideration of each response separately.

(J. J. Bartko)

427

AZORÍN, F. (Inst. Statist. Investigations, Madrid) **6.0** (8.7)

Some statistical problems in the construction of consumption scales—*In Spanish*

Trab. Estadíst. (1959) **10**, 63-73 (11 refs.)

A mainly expository paper on consumption scales and related statistical questions. The author considers consumption scales as a special case of the general problem of measuring clusters or composite units when simpler units are not clearly defined. An attempt is made to define such units in order to measure clusters, trying to make the definition practically efficient from the point of view of information and cost.

In the special case of consumption (in economics) reference is made to the previous work of Woodbury and others. In relation to food consumption and the corresponding scales based in requirements, the lack of a general acceptance of levels as to specific necessities is emphasised. For example, as to the requirements of proteins of animal and vegetable origin.

A brief consideration is given to the origin of Engel curves as the relation between the expenditure or consumption of some item and the income or total expenditure in the families of a certain population. The work of Sydenstricker and King is reviewed as well as their terminology and concepts based on the use of " ammains " and similar units.

A more detailed exposition is made of the method due to Friedman who takes, as the unit of scale, the married couple instead of the adult male because of improved information and less irregularity in the distribution of budgets. The Friedman method based on the fitting of a line to variables made independent of the number of consumption units is discussed—also the advantages of considering a continuous variation of consumption with the age, within each sex. A disadvantage of this method is the use of the same scale in the computation for the item considered and for the total expenditure. However, when a weighted average of the units for each item is taken as the unit for the total, the method loses its simplicity. A different approach is used by Houthaker and Prais. They state the Engel function in a form in which the independent variable is the income (or total expenditure) per unit and the dependent one the same as in the method by Friedman. Making two different assumptions about the form of the function, the values of the units can be computed by a successive approximation procedure. A different type of function could be used—as pointed by the author who also stresses the practical convenience of the method used by Brown who makes a previous classification of families into types of constant composition.

Finally, attention is given to the possibilities of extension to different fields, such as productivity.

(S. Rios)

428

DALL'AGLIO, G. (University of Rome) **6.0** (–.–)

On the compatibility of two-dimensional distribution functions—*In Italian*
R.C. Mat. Univ. Roma (1959) **18**, 385-413 (7 refs., 3 tables, 3 figs.)

Three two-dimensional distributions, respectively of the pairs of variables (x, y), (x, z), (y, z), are said to be compatible (" compatibili ") if a three-dimensional distribution exists, of which they are the two-dimensional marginal distributions. This paper studies the conditions for the compatibility, related to the distribution functions.

After giving a necessary condition, the author studies the class C of the distribution functions of the couple (y, z) compatible with two given two-dimensional distributions of (x, y) and (x, z). Under the evidently necessary condition that the two given distributions have the same marginal distribution for the variable x, the class C is not empty, and the following statements hold:

(i) C is a sub-class of the " Fréchet class " of the two-dimensional distribution functions which have as marginal distributions the distributions of the variables y and z.

(ii) The class C has a maximum and minimum function (maximum and minimum in every point). There are, however, distribution functions between the minimum and the maximum ones which do not belong to C.

(iii) If two functions belong to the sub-class C, all their linear combinations with non-negative coefficients having a unit sum are also in class C.

In the second part of the paper, the class C is studied when the associations between x and y and between x and z are of particular types. In the case of independence, the class C evidently attains its maximum, i.e. it coincides with the " Fréchet class ". This can occur also if y and z are not independent of x. If both y and z are connected to x by a distribution function, maximum in the Fréchet class, the distribution function of (y, z), maximum in the Fréchet class, belong to C, and it is only the element of C under some large conditions.

It is sufficient, for instance, that the distribution function of x be continuous. The same results occur if both the two given two-dimensional distribution functions are minimum: the maximum function belongs to C, and is the only element of C under analogous conditions.

In the last part of the paper the results found are used in order to determine the minimum and maximum values, in C, of the moments about the line $y = x$. Later, the minimum and maximum values of the correlation coefficient r are found, i.e. the extreme values of the correlation coefficient between two variables, when each of them is connected to a third variable by a known law.

(G. Dall'aglio)

DAVID, H. A. & ARENS, B. E. (Virginia Poly. Inst., Blacksburg) **6.1** (0.2)

Optimal spacing in regression analysis—*In English*
Ann. Math. Statist. (1959) **30**, 1072-1081 (8 refs., 2 tables, 1 fig.)

The bulk of this paper is concerned with the following problem: suppose y is a quadratic function of x, where x is a variable that is controllable without error and y is observable with constant variance σ^2. It has been decided to approximate y as a linear function of x, perhaps because of knowledge that the quadratic component is small. The problem is to select the values of x so that the observations (x_i, y_i), in an experiment with a fixed number of observations, will provide the " best " estimate of the actual y value.

The authors first discuss this problem in general, where y is an arbitrary function of x. In terms of the true dependence of y on x, conditions for optimal spacing of x for a given approximating function are derived. They treat the situation where the observations y are made without error, and obtain two different sets of x values for optimality. One corresponds to the condition of closest overall fit, and the other to the condition of minimising the maximum deviation; both are satisfied only approximately. With this background established, the remainder of the paper is devoted to the particular problem already mentioned.

In the problem of the quadratic being fitted with a straight line, the authors show how the optimum position can be achieved with only two values of x, and then proceed to demonstrate that these two locations, for either criteria of the optimum position, are a function of the ratio σ' to the true quadratic coefficient in y, where $\sigma'^2 = 2\sigma^2/n$, n being the total number of observations. A table is given showing the dependence of the x-locations on this ratio. A second table is given which compares the goodness-of-fit for the optimally determined x-values with four other choices, this being done for each criterion of the optimum position. The authors conclude with a brief numerical example.

(D. H. Shaffer)

GUTTMAN, L. (Israel Inst. Applied Social Res. and Hebrew Univ., Jerusalem) **6.5** (6.3)
Metricising rank-ordered or unordered data for a linear factor analysis—*In English*
Sankhyā (1959) **21**, 257-267 (19 refs.)

This paper is concerned with Thurstone's problem of the factor analysis of observed rank-orders with respect to having also the underlying factors only rank ordered. That is, for a population P of subjects and a set $\mathfrak{J}$ of tests, the metrics of the observed scores are arbitrary up to monotonic transformations within each test. The problem is to find those scores which will yield a correlation matrix possessing properties that are most desirable from some point of view of common-factor analysis and which preserve the observed rank-orders within tests.

The author begins with a brief exposition of the approach of other investigators, and then he proceeds to solve the general problem of assigning real numbers to given qualitiatve categories for a given population P in such a way that the resulting numerical variables have linear regressions on each other. The bivariate case is treated first and it is shown that if the original scores already have linear regressions, there is no profit in trying to transform them into new scores. Extensions are made to the multivariate situation.

In general, it is shown that at most one set of new scores can be found to maintain the observed rank-orders. The factor analyst has no freedom to mould the new correlation matrix by further consideration. The paper also shows how to compute all scoring systems which will yield linear regressions, the new scores possibly having polytonic as well as monotonic relations with the original rank order. The analysis presented does account completely for the dependence of tests on each other. The complete sets of regression linearising scores always completely reproduce the observed pairwise occurrences, or completely " explain " any bivariate dependence.

(J. C. Reed)

GUTTMAN. L. & GUTTMAN, RUTH (Hebrew Univ. ; Hadassah Med. School, Jerusalem) **6.7** (4.7)
An illustration of the use of stochastic approximation—*In English*
Biometrics (1959) **15**, 551-559 (4 refs., 2 tables).

The paper presents an application of the Robbins-Monro [*Ann. Math. Statist.* (1956) **27**, 879-886] procedure for stochastic approximation, used in regression situations where the regression is monotonic. The procedure estimates the independent variable x^* for which the dependent variable takes on a preassigned value, y^*. The method is distribution-free; the only assumption made in this experiment was that the regression was monotonically increasing.

The experiment was of a sequential type. For the given value, y^*, of the dependent variable, successive values, obtained by formula, were assigned to the independent variable, x. This formula is the key to the entire procedure: $x_{n+1} = x_n + a_n(y^* - y_n)$.

Apart from a reasonable first guess x_1, which is not crucial, the primary practical problem involved is determining the weight, a_n. Robbins and Monro proved that if the a_n are proportional to the series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$, then the successive estimates of x^* corresponding to y^* will converge. While any constant of proportionality insures convergence, a wise choice hastens the convergence.

After the process has been applied, say, n times, a further repetition of the procedure fails to produce significant departure from the value of x^* obtained in trial n. It is pointed out that there is no statistical rule for stopping at some finite n with an associated confidence interval for x^*.

In addition to the desired value of x^* obtained, some scanty information concerning the regression curve itself is obtained from the points plotted in the vicinity of y^*.

While the Robbins-Monro procedure is only applicable to monotone regression situations, other stochastic approximation procedures are available; e.g., the Kiefer-Wolfowitz method [*Ann. Math. Statist.* (1952) **23**, 462-466] for estimating a unique maximum.

It was assumed that the regression was flat up to a certain point, after which the regression increased. The authors were quite interested in determining the point at which the regression curve ceased being flat and began to rise. However, no procedure for obtaining this point is available.

The dependent variable (y) was a ratio of treatment-over-control cell counts in a biological experiment. The ratio was used in place of a difference because it permitted a better guess at the constant of proportionality, a_n. It is worth noting that should division by zero arise in using the ratio, the flexibility of stochastic approximation freely permits shifting to a difference statistic. Two examples of the procedure are given.

(H. R. Baird)

HARRIS, E. K. (Robert A. Taft Sanitary Engineering Center, Cincinnati) **6.8** (4.4)
Confidence limits for the LD_{50} using the moving average-angle method—*In English*
Biometrics (1959) **15**, 424-432 (8 refs., 2 tables)

Moving average interpolation for LD_{50} estimates has been proposed by Thompson [*Bact. Rev.* (1947) **11**, 115-145] and extended by Bennett [*J. Hyg., Camb.* (1952) **50**, 157-164] to include the angular transformation. Formulas and tables useful in calculating approximate confidence limits for the LD_{50} without reference to the angular transformation are also available.

The present paper outlines methods for determining " exact " confidence limits and for making significance tests under the moving average-angle procedure. The confidence limits are not strictly exact since the observed rather than the expected proportional response is transformed to an angle. The author limits discussion to equally spaced doses and equal numbers of animals exposed to each dose but states that extension to differing intervals and/or numbers of animals is easily accomplished.

The method consists of applying the arcsine transformation to the observed proportional responses, using Bartlett's adjustments when the proportion is 0 or 1. Simple averages of k successive angles are computed and each average angle associated with the middle dose of the respective set of k doses (usually 3 or 5). The LD_{50} is estimated by linear interpolation between the successive doses whose average angles bracket 45°.

Defining A as the fraction by which the dosage increment is multiplied in the interpolation process, and under the assumption of fixed doses and normal distribution of the average angles, confidence limits for the LD_{50} are based on limits for A determined from Fieller's theorem. Tables list A_L and A_U for $A = 0(0{\cdot}05)0{\cdot}50$ and $g = 0{\cdot}01(0{\cdot}01)0{\cdot}10$; $0{\cdot}10(0{\cdot}05)0{\cdot}80$ for $k = 3$ and 5, where $g = (Z^2 \operatorname{Var} Y)/Y^2$; Z is the normal deviate corresponding to the two-side confidence interval and Y is the difference between the bracketing average angles. Extensions are indicated for A greater than 0·5 and the author states that more complete tables are available on request.

A test of significance is given for the difference between two estimated LD_{50}'s, based on the assumption that the dosage response curves are essentially parallel. Mathematically the difference between two estimates is expressible as the sum of the difference of the pertinent doses and the ratio of the difference of the corresponding average angles to the common slope of the dosage response curves. Under the assumption that both quantities in the ratio are normally and independently distributed, a lower confidence limit is computed for the ratio. The LD_{50} estimates are judged significantly different if, when the lower confidence limit on the ratio is substituted for the ratio itself in the expression for the difference of the LD_{50}'s, the result is greater than or equal to zero. A numerical example is given to illustrate the method.

(M. F. Peck)

433

KRUSKAL, W. H. (University of Chicago) **6.9** (6.5)
Ordinal measures of association—*In English*
J. Amer. Statist. Ass. (1958) **53**, 814-861 (93 refs., 1 table, 15 figs.)

The purpose of the author in this review paper is to discuss probabilistic or operational interpretations of the population values of several measures of association of bivariate populations. The author is especially concerned with ordinally invariant (rank) measures and the structure of the bivariate population is not generally specified.

As a background, the author first discusses the standard correlation ratio and correlation coefficient. The first ordinally invariant measure given is " quadrant association ", to which the author assigns the archaic Greek letter " *koppa*." Following this are introductory sections on Kendall's " *tau* " and Spearman's " *rho* ". In each case a probabilistic interpretation of the measure for the population is provided. It is notable that for the definition of *koppa* the probabilities of certain relationships among the elements (X, Y) of a bivariate random variable are required. For the definition of *tau* pairs of identically distributed independent bivariate random variables are considered; while for *rho*, triples of such variables are needed.

Relationships among *koppa*, *tau* and *rho* are considered. Inequalities involving each pair of measures are given and all but one of these bounds are shown to be the best possible. Values of the three measures are given for the bivariate normal distribution (in terms of the correlation coefficient) and for a one-parameter family of distributions uniformly distributed over portions of the unit square. The author points out that the three measures are comparable in that they all measure " concordance " (as opposed to " connection "), they all range from -1 to $+1$ and independence of X and Y is a sufficient condition that each measure be zero.

Sample estimators of all three measures are given, including the more complicated unbiased estimator of *rho* as well as the usual biased estimator. Variances and asymptotically unit normal functions of the estimators are provided. References to exact sampling distributions are given. The author discusses relative merits of the estimators and in particular points out that they differ in their ability to discriminate against various alternatives to independence.

The measures are extended to general distributions, considering wholly discrete cases, i.e., cross-classification, as well as the continuous form without tied items. Relationships among the measures in the two cases are discussed. A historical survey of ordinal measures of association is given in the final section: the list of references is encyclopædic.

(S. Krane)

434

LANCASTER, H. O. (University of Sydney) **6.2** (2.1)

Zero correlation and independence—*In English*

Aust. J. Statist. (1959) **1**, 53-56 (1 ref.)

The author considers the conditions for which zero correlation between two random variables implies their mutual independence.

The following theorems are proved:

Theorem 1. If the random variables $x_1, \ldots x_n$ are jointly normally distributed $N(O, V)$, where their covariance matrix **V** is positive definite, then the necessary and sufficient condition for the independence of two linear forms, $l_1 = \underset{i}{\Sigma} a_i x_i$ and $l_2 = \underset{i}{\Sigma} b_i x_i$, is that they be uncorrelated.

Theorem 2. A necessary and sufficient condition for the mutual independence of the random variables x and y is that every function of x with finite variance is uncorrelated with every function of y with finite variance.

Theorem 3. If $\{x^{(i)}\}$, $\{y^{(j)}\}$ are complete sets of orthonormal functions defined on the marginal distribution functions $G(x)$, $H(y)$ of a joint bivariate distribution $F(x, y)$, then a necessary and sufficient condition for independence of the distributions of x and y is that $\mathscr{E}(x^{(i)}y^{(j)}) = \rho_{ij} = 0$ for every pair of positive integers i, j.

(J. Gani)

MORAN, P. A. P. (Australian National University, Canberra) **6.0** (5.1)

Regression, correlation and structure—*In English*

Aust. Meteorological Mag. (1959) **24**, 1-9 (4 refs.)

This paper sets out the basic statistical theory of the association between two or more measured quantities. The distinction is made between a mathematical variable and a random variable or variate, and by a simple example the concept of the " power " of a significance test is explained.

The analysis of association is necessarily carried out by the use of models and the theory involved in each of four such models is explained. The models are:

(i) the regression model which contains only one random variable (y_i) whose mean is linearly dependent on a mathematical variable (x_i), and the power of the associated significance test is increased by examining the means of the y_i's at widely separated values of x.

(ii) The bivariate normal distribution model which contains two interdependent variates. Distinction is made between the regression line considered as a property of the bivariate distribution and alternatively as part of the definition of the regression model.

(iii) The " errors in variables " model in which two random variables are measurements involving random errors, and the model is used to arrive at an estimate of the underlying linear relationship between the unobserved structural variables which may be random or not.

(iv) Berkson's model, in which analysis is made of the relationship between two quantities. One of these is an observed random variable whose means are dependent linearly on unobserved random variables whose means, in turn, are known specified constants.

(P. A. P. Moran)

NASS, C. A. G. (Netherlands Inst. Preventive Medicine, Leiden) **6.9** (5.2)

The χ^2 test for small expectations in contingency tables with special reference to accidents and absenteeism—*In English*

Biometrika (1959) **46**, 365-385 (13 refs., 12 tables)

When M men are classified with respect to the numbers of their accidents (or absences) in n periods (of perhaps varying lengths) the frequencies are arranged in an n-row by an m-column contingency table. The hypothesis of independence is that (relative to the various degrees of their proneness) each man presents the same pattern of absences over the periods. By the nature of the data the " expected " cell frequencies may be unity or less and this paper investigates, numerically, the behaviour of the chi-squared test function (here called G) particularly in relation to the approximation obtained by fitting a Pearson type III curve to the first two moments of G as given by Haldane [*Biometrika* (1937) **29**, 133-143 and *Biometrika* (1939) **31**, 346-355].

After some discussion of various special cases of the situation where the probabilities in one margin are known (with only one row in particular, and even more particularly where in this case the expected frequencies in the cells here, according to the null hypothesis, are all equal) the actual distributions of G are enumerated for

(*a*) two columns (with totals both 20) and 25 columns (10 totalling to 1, 15 to 2), and

(*b*) 2 columns (totals 20 and 280) and 200 columns (100 each totalling 1 and 2).

The type III distribution is found to give a good approximation to the 5 per cent. point in both cases.

The paper concludes with four applications of the test to data on accidents and absenteeism. This is followed by a discussion chiefly concerned with the interpretation of significantly large values in this field of application.

(D. E. Barton)

OSTLE, B. & STECK, G. P. (Sandia Corporation, Albuquerque) **6.2** (3.6)

Correlation between sample means and sample ranges—*In English*

J. Amer. Statist. Ass. (1959) **54**, 465-471 (3 refs.)

The correlation between the mean and the range is investigated for sampling from symmetric and asymmetric populations. It is shown that if a random variable is symmetrically distributed with finite variance, then the mean and range of a random sample are uncorrelated. However, as is demonstrated by a counter example, zero correlation does not imply symmetry. A necessary and sufficient condition that the correlation between the mean and the range be positive (negative) is presented. It is also proved that symmetry of the parent population implies that the sample range and midrange are uncorrelated.

(S. Addelman)

PANIZZON, G. (School of Statistics, Padua) **6.1** (6.2)

A generalised multiple linear correlation coefficient—*In Italian*
G. Economisti (1959) **18** (New series), 81-101 (14 refs., 4 tables)

The author considers measures of the correlation among several variables and in particlar the indexes proposed by Hotelling, namely: the canonical correlation index and the vector correlation index. Following the approach of Levis [*R. C. Mat. Univ. Roma* (1958) **17**, 423-428] the author starts from a geometric approach to the problem of the relations between two groups of variates and calculates the coefficient r as a geometric mean of the regression parameters.

After having considered the properties of the generalised multiple linear correlation coefficient, he demonstrates that it is equal to the vector correlation index.

He completes his exposition with an application to anthropometric measures, the correlation matrices are calculated together with the generalised correlation coefficient.

(P. Bandettini)

Editorial Note: see also abstract No. 85, **6.1.**

PAWLIK, K. (Psychological Inst., University of Vienna) **6.9** (–.–)

The maximum coefficient of contingency when the numbers of rows and columns are unequal—*In German*
Metrika (1959) **2,** 150-166 (10 refs.)

The contents of this article may be described shortly by a quotation of Cramér: " The mean square contingency . . . is . . .

$$\Phi^2 = \sum_{i,k} \frac{(p_{ik} - p_{i\cdot}p_{\cdot k})^2}{p_{i\cdot}p_{\cdot k}} = \sum_{i,k} \frac{p_{ik}^2}{p_{i\cdot}p_{\cdot k}} - \Lambda \text{ "}$$

(p_{ik} = probability that an individual belongs to the ith row and kth column)

" . . . by means of the inequalities $p_{ik} \leqq p_{i\cdot}$ and $p_{ik} \leqq p_{\cdot k}$ it follows from the last expression that $\Phi^2 \leqq q - \Lambda$, where $q = \text{Min}(m, n)$ denotes the smaller of the numbers m and n, or their common value, if both are equal. Further, the sign of equality holds in the last relation if, and only if, one of the variables is a uniquely determined function of the other. Thus

$$0 \leqq \frac{\Phi}{q - \Lambda} \leqq \Lambda \text{ . . .."}$$

[see p. 282 of *Mathematical Methods of Statistics* (1946)].

Instead of using Φ^2 which is defined by

$$CC = \sqrt{\frac{\chi^2}{N + \chi^2}} = \sqrt{\frac{f^2}{\Lambda + f^2}}$$

where $f^2 = \sum_{i,k} \frac{\tilde{p}_{ik}^2}{\tilde{p}_{i\cdot}\tilde{p}_{\cdot k}} - 1$, when $\tilde{p}_{ik} = \frac{\nu_{ik}}{N}$ and ν_{ik} is the number of individuals in the place (i, k), N the total number of individuals. Therefore f^2 is the sample characteristic corresponding to Φ^2.

It is shown that the results must be the same as cited above. The proof given is not exact in the mathematical sense, but is substantially true and may be made rigorous without difficulty.

The author finally discusses a " measure of correlation " in a contingency table as introduced by Gebelein. Since Gebelein starts from a quite different point of view, there is no connection between the two methods.

(K. Weichselberger)

Remark of the Regional Editor:

The correction of several misprints in this paper appear in *Metrika* (1959) **2,** 256.

(J. Pfanzagl)

SIBUYA, M. & HAGA, T. (Inst. Statist. Math. and Sano Pulp Co. Ltd., Tokyo) **6.1** (11.1)
Orthogonal polynomials without constant term—*In English*
Ann. Inst. Statist. Math., Tokyo (1959) **10**, 209-222 (2 refs., 2 tables)

The problem of fitting curves which pass through the origin requires the numerical values of orthogonal polynomials which have no constant terms, in the same way as the problem of fitting general curves requires Tchebysheff-Fisher polynomials. With this in mind, the authors give expressions of such polynomials and tabulate the numerical values related to them.

Let N be a positive integer and $X_n^{(N)}(x)$ $(n = 1, 2, \ldots, N)$ be polynomials in x of degree n whose constant terms vanish. These polynomials are required to have the orthogonal property with respect to N equidistant points; that is, the summation over $x = 1, 2, \ldots, N$ of $X_n^{(N)}(x)X_m^{(N)}(x)$ is required to vanish for any distinct two integers n and m. This property determines $X_n^{(N)}(x)$ up to constant multipliers. Using some quantities calculated with the aid of these polynomials, the coefficients of a regression model which is expressed in the form of the sum of a polynomial of degree n and a homogeneous error term are estimated in the sense of least square. The quantities required for this purpose are the values of orthogonal polynomials $X_n^{(N)}(x)$ and their norm $S_n^{(N)}$—the square sum of $X_i^{(N)}(x)$ over $x = 1, 2, \ldots, N$.

The explicit forms of $X_n^{(N)}(x)$ and $S_n^{(N)}$ for $n = 1, 2, 3, 4$ are given: for general n, the recursive formulae of $X_n^{(N)}(x)$ are given. $X_n^{(N)}(x)$ is also expressed as the nth difference of a certain polynomial of degree $2n$. The numerical values $X_n^{(N)}(x)$ for $n = 1, 2, 3, 4$; $N = 2(1)23$, $x = 1(1)N$, with the values of $S_n^{(N)}$ are tabulated in Table 1, and the coefficients of $X_n^{(N)}(x)$ in Table 2.

The authors also consider the heterogeneous case where the standard deviation of the error term in the regression model is proportional to x, which can be reduced to the homogeneous case.

(K. Isii)

TEGHEM, J. (Université Libre de Bruxelles) **6.4** (4.8)
Confluence analysis—*In French*
Cahiers du Centre d'Études de Rech. Operat. (1959) **3**, 5-34

The purpose of this paper is to show the object and methods of confluence analysis, with as little technical detail as possible.

Using Markoff's theorem, the author first recalls the mathematical model to which the least squares method may be applied in the problem of estimating a structural linear equation. Among other things, it is assumed that there is simple structural collinearity and absence of errors upon the explanatory variates.

As Frisch emphasised, the joined failure to obey these two hypotheses may lead to fictitious determinations, without sense, if the least squares method is applied. The author develops these remarks by substituting Koopmans' probabilistic language for Frisch's non-probabilistic one [Koopmans, *Linear regression analysis of economic time series*, Netherlands Economic Institute (1937) Haarlem]. He states the model used by Koopmans in this last work. In this model, all variates are liable to errors, but simple structural collinearity is still assumed, so that Frisch's criticisms concerning multicollinearity remain valid. When there are no errors multicollinearity would clearly appear in the estimations by the determination of the latter. The presence of errors conceals this indetermination as errors have what Frisch calls a " cushion effect ". The case of simple collinearity close to multicollinearity, are also sensitive to the " cushion effect ".

The author then shows how Frisch achieves the following three aims: (i) to test simple collinearity, (ii) to obtain spaces answering simple collinearity, and (iii) to estimate, in each of these spaces, the coefficients of the corresponding structural equation. He recalls Frisch's non-probabilistic model, infers fundamental properties and explains the method. The simple collinearity test and the obtaining of spaces answering this collinearity are based upon subjective comparison, for each pair of variates (x_i, x_j), of the regression coefficients of x_i on x_j in the various spaces to which the pair belongs. In relation to every couple, the other variates are classified into useful variates, superfluous variates and determinental variates. Each of the possible spaces of the variates is then studied in its whole, according to the nature of the variates with regard to the couples of the space. These comparisons and this study are made with the help of numerous graphs and tables, the bunch maps and the star maps.

Only those spaces are retained in which all variates are useful with regard to all the couples of other variates in the space. In them and in them only, is set the problem of estimation. Frisch solves this problem by using his diagonal regression. The lack of precision of the estimate is expressed by a factor of significance.

(J. Teghem)

UEMATU, T. (Inst. Statist. Math., Tokyo) **6.4** (0.1)

Note on the numerical computation in the discrimination problem—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **10**, 131-135 (3 refs.)

The author attempts to simplify the numerical computation of the secular equation, the solution of which is needed in the discrimination problem. This secular equation has, on each side of its equality, two matrices: the one is positive definite, and the other is symmetric and singular. The usual computation method for solving the equation contains the computation of the inverse matrix of the positive definite matrix and its multiplication with the other matrix. By these operations the equation is reduced to the usual form and solved.

In the actual discrimination problems, especially in the quantification problem, the dimension of the equation concerned is usually very high, so that the above-mentioned operations of matrices are very troublesome. In the development of this situation some simpler methods are presented. In most practical cases the number of groups, into which the discrimination is carried out, is very small. The rank of the above-mentioned singular matrix is less than the number of groups, so that it is very small. More precisely, this singular matrix is given by the product of some matrix with the transposition of it, the number of rows of the matrix being not greater than the number of groups. This makes it possible to reduce the solution of the original equation to the solution of some secular equation with very small dimension. This reduction simplifies the numerical solution of the equation.

(Y. Suzuki)

WEICHSELBERGER, K. (Sozialforschungsstelle, Universität Münster) **6.9** (4.8)

The estimation of parameters for contingency tables with marginal totals given in advance (I)—*In German*

Metrika (1959) **2**, 100-130 (4 refs.)

The paper deals with the problem of estimating cell frequencies of contingency tables, when the exact totals of rows and columns are known in advance and a sample is taken to obtain information on the cross-classification. As the totals of rows and columns in the sample are not usually in accordance with the data given in advance the question of adjustment arises.

Results are obtained for the case of an infinite universe. In this case the estimation of the cell-probabilities p_{ik} is bounded by the marginal-probabilities given. Maximum likelihood estimators are obtained as $\hat{p}_{ik} = h_{ik}/c_i + d_k$, where h_{ik} is the relative frequency in the sample and the c_i and d_k are Lagrange-multipliers which are determined by the side-conditions giving the relation to the marginal probabilities. It is shown that this system of equations has one and only one solution with all $\hat{p}_{ik} \geqq 0$, provided there does not exist a row i_0 so that $h_{i_0 k} = 0$, for all k, for which $h_{ik} > 0$ in any other row. If this condition is not fulfilled, there always exist $p_{i.}$ and $p_{.k}$, so that the system has no solution. But it is possible, of course, to use $x_{ik}+1$, instead of x_{ik} in these cases, without changing the asymptotic properties of the maximum likelihood solution.

The proof given is synthetic, i.e., a solution is constructed by successive approximation. Finally it is shown that the solution is unique.

(J. Pfanzagl)

Editorial Note: see also abstract No. 445, **6.9.** Weichselberger

WEICHSELBERGER, K. (Sozialforschungsstelle, Universität Münster) **6.9** (4.8)
The estimation of parameters for contingency tables, with marginal totals given in advance (II)—*In German*
Metrika (1959) **2**, 198-229 (12 refs.)

At first, the possibilities of solving the system of maximum likelihood equations obtained in the first part of the article is discussed.

The method of successive approximations is used to calculate the solutions. The elements of the approximating sequences fulfil the relation $\hat{p}_{ik} = h_{ik}/c_i + d_k$, but do not equal both row totals and column totals. One step approximately fits the row totals, the next the column totals, and so on. The convergence is proved using the main result obtained in the first part of the paper.

Three different methods fitting the totals approximately are discussed; they are different in the complexity of the single steps and in the rate of convergence. Their qualities are illustrated and tested by means of an example.

Deming and Stephan have proposed a solution to the problem of contingency tables with fixed marginal totals by means of a least squares adjustment. The method leads to a system of equations $\bar{p}_{ik} = h_{ik}(1+\gamma_i+\delta_k)$ together with the conditions concerning the marginal probabilities. The solutions of this system are discussed; they are found to differ from the maximum likelihood solution only by terms of second and higher order in $(p_{i.}-h_{i.})$ and $(p_{.k}-h_{.k})$.

All proofs of convergence concerning the calculation by iterative adjustment in this case can be given by means of the results of Part I.

Finally, the method of iterative proportional adjustment —which constitutes a third kind of solution—is discussed. It leads to $\bar{\bar{p}}_{ik} = h_{ik}C_iD_k$.

The solution of this system can also be calculated by iterative methods and the proof given in Part I remains valid. The solution is also of theoretical interest, as it can be shown that it uses only the information contained in the sample without regard to the marginal probabilities. Under certain rather general conditions the distribution in a contingency table is determined by two sets of parameters. One of them gives the marginal totals, the other one gives the remaining information, i.e. the association within the table. The method of iterative proportional adjustment combines the second set of parameters estimated from the sample with the totals given in advance. The author states that this solution is to be preferred, if it is suspected that there is a bias in the totals of the sample.

(K. Weichselberger)

Editorial Note: see also abstract No. 444, **6.9.** Weichselberger

445

WETTE, R. (Zoolog. Inst., Universität Heidelberg) **6.0** (6.1)
Analysis of regression and causal relation in biology—*In German*
Metrika (1959) **2**, 131-137 (27 refs.)

A concise presentation of problems in the analysis of relations between biological variables is given, together with many references to the pertinent literature up to 1956—the time when the paper was originally read by the author. A clear distinction is made between two fundamentally different problems in biometry.

The relation between two or more variables can be adequately analysed by the standard methods of regression and correlation analysis, if and only if the object is a mere description of the relations between observational data. Different methods, however, have to be applied if the structural relation is at issue, i.e., the relation which is postulated to cause the empirical relation as observed in a sample. Whereas there is only one single structural relation between two biological variables, ordinary regression calculation yields two regression lines. Consequently the parameters estimated by this method are not identifiable with the parameters of the structural relation.

Various problems are discussed briefly and an outline is given of appropriate methods in order to arrive at consistent estimates of the linear structural relation between two biological variables. The case of both variables being subject to error is also considered. The review of the literature on each of the problems discussed will be appreciated by those who desire further information on the subject.

(F. X. Wohlzogen)

446

DAVID, FLORENCE N. (University College, London) **7.5** (3.1)

The z-test and symmetrically distributed random variables—*In English*
Biometrika (1959) **46**, 123-129 (6 refs.)

Some years ago David and Johnson [*Ann. Math. Statist.* (1951) **22**, 382-392] gave the exact cumulants and cross-cumulants of the distributions of the sums of squares, usually associated as test-criteria for the general linear hypothesis, under extremely wide assumptions. It had been intended to use these exact values to explore arithmetically the effect on the F (or z) test of departures from the basic assumptions of the analysis of variance, but the amount of arithmetic involved was intimidating. In this present paper, techniques are described which considerably reduce the exploration of departure from normality etc., as far as the calculations are concerned. The case of symmetrically distributed variables in the one-way classification is the only one discussed, but it is pointed out that the techniques are applicable in any of the analysis of variance set-ups which are special cases of the general linear hypothesis.

For normal variables, $2z = \log_e F$ is approximately normally distributed for degrees of freedom which are not small and even for small degrees of freedom can be reasonably represented by the Edgeworth form of the Gram-Charlier series. It may be assumed when the variables are not normal but the divergences are not too marked that the distribution of z may still be represented by this series. Writing z_n for z derived from a normal population, and z_A for that from a non-normal population, a series expansion is obtained for z_A about a given significance level of z_n. It is shown that for moderate divergences in the parent population the effect on the nominal significance level is not large.

The case of unequal group sizes is also discussed. It is pointed out that the effect of unequal group sizes is to make the mean and variance of z_A more nearly those of z_n and that calculations for the effect of non-normality on the distribution of z are therefore best carried out under the assumption that all the groups are of equal size.

The inverse series of Cornish-Fisher is used to obtain comparisons between z_A and z_n at the α per cent. level of significance. The difference between the two values for a given level of significance is shown to be of small order and substantially in agreement with that given by Gayen [*Biometrika* (1950) **37**, 236-255].

(Florence N. David)

HALL, W. J. (Univ. of North Carolina, Chapel Hill) **7.7** (5.4)

The most-economical character of some Bechhofer and Sobel decision rules—*In English*
Ann. Math. Statist. (1959) **30**, 964-969 (10 refs.)

Bechhofer has considered a single-sample multiple-decision procedure for choosing, among a group of normal populations with common known variances, that population with the largest mean; and, with Sobel, a procedure for choosing the normal population with the smallest variance. Several other analogous problems have also been considered. [*Biometrics* (1958) **14**, 408-429.] They suggest, with only intuitive justification, choosing the population with the largest (smallest) sample mean (variance), and give tables for finding the minimum sample size (assumed equal for all populations) which will guarantee a correct decision with prescribed probability when the extreme population parameter is sufficiently distinct from the others. This paper gives justification for a wide class of such procedures, proving that no other rules can meet this guarantee with a smaller (fixed) sample size; that is, such rules are most economical [*Ann. Math. Statist.* (1958) **29**, 1079-1094]. Proof of the most-economical character of these rules is achieved by proving their minimax character when a suitable loss function is introduced.

The main result is applicable to any analogous problem of choosing the population with the most extreme parameter when, for each sample, there is a sufficient numerical statistic with a monotone likelihood ratio and the (numerical) parameter is a location or scale (but not range) parameter in the distribution of the statistic. It is extended to parameters which are not location or scale parameters, but the guarantee then holds only at a specified location; for many problems, a least favourable location can be determined so that the guarantee can be made to hold irrespective of location. For example, the procedures of Sobel & Huyett [*Bell Syst. Tech. J.* (1957) **36**, 537-576] for choosing the largest of several binomial parameters are proved to be most economical. The broader optimum properties of these latter procedures, for choosing the population with the largest "survival probability" irrespective of the form of the distribution, is also discussed. Extensions to problems of ranking the populations according to the parameter values, or of grouping them by ranks, are also given.

(W. J. Hall)

Editorial note: see abstract No. 96, **7.7.**
Hall

HENDERSON, C. R., KEMPTHORNE, O., SEARLE, S. R., & von KROSIGK, C. M. 7.3 (7.4)
(Cornell Univ. & Iowa State Coll.)
The estimation of environmental and genetic trends from records subject to culling—*In English*
Biometrics (1959) **15**, 192-218 (5 refs., 3 tables)

The formal application of least squares to the estimation of parameters in the presence of selection leads to biased estimates; the extent and direction of this bias depends on the method of analysis employed. The authors discuss this situation for a special example, i.e. the estimation of productivity trends due to heredity and environment, in the presence of culling. They propose two more appropriate methods for the analysis of such data. Both methods regard as fixed effects a " general mean production ability " and the " environmental effect of the kth year ". In the first method a " group effect " of cows-in-a-herd is added in which groups may constitute daughters of the same bull, groups entering the herd at the same time, etc. The other method specifies this " group effect " as " the amount by which the group of individuals entering the herd in year t deviate." Both methods regard the hereditary effect, the " producing ability of a cow," as a random variable with zero mean and equal variance σ_c^2, and independent of the error (variance σ_e^2). In the first method (Henderson), the " repeatability " index, $r = \sigma_c^2/(\sigma_c^2+\sigma_e^2)$ is assumed to be known, and thus the normal equations obtained, for the maximum likelihood estimation of fixed effects, have the familiar form of a mixed model analysis. Consequently, the estimation procedure is quite straightforward; it is described in detail and illustrated with a hypothetical set of data. The second method (Kempthorne and von Krosigk) assumes that r is not known and must be estimated from the data. The maximum likelihood method in this case leads to a complicated set of equations which, the authors suggest, should be solved by the Newton method for several variables, with perhaps certain simplifications similar to those used in bio-assay. The presentation of the maximum likelihood solution in this method should be of special interest to proponents of the " recovery " analysis. The juxtaposition of the two methods shows, not only in this case, how the simple analysis of a mixed model, under the (unrealistic) assumption that the population variances are known, must be replaced by a quite complicated analysis if the two types of variances are to be estimated. The second method is illustrated by data of the Iowa Board of Control herd at Woodward. In the final section, the fourth author (Searle) shows the obvious equivalence of the two methods when r is known. Henderson uses two types of derivations (general linear hypothesis and " step-down "), whereas Kempthorne and von Krosigk use the " step-down " technique.

(R. E. Bargmann)

449

MITRA, S. K. (Indian Statistical Institute, Calcutta) 7.4 (–.–)
Some remarks on the missing plot analysis—*In English*
Sankhyā (1959) **21**, 337-344 (9 refs., 4 tables)

Several authors (Neyman, Welch, Pitman, Kempthorne, Wilk and others), while investigating the possibility of validating the customary F-test in the analysis of variance for designed experiments, have considered the stochastic character of the observed variables as due primarily to the random assignment of the treatments to the experimental units. It is possible (theoretically, at least) to write down their joint distribution as soon as the randomisation procedure R is specified. Their researches tend to show that the F-test is unbiased in the sense that under the null hypothesis, both the numerator in F (the treatment mean-square) and the denominator (the error mean-square) have the same expected value.

The object of the present paper is to demonstrate that this is no longer true with the analysis-of-variance F test when the yields on some of the units are missing. For simplicity, only the cases of the randomised block and latin square experiments with a single missing observation are discussed. For computing the expected values, independent repetitions of R (with the same set of experimental units reporting missing yields each time) are considered.

The author remarks that such a bias possibly affects the test of equality of treatment effects only in so far as the use of the percentage points of the F-distribution for judging the significance of the computed F will either under-estimate or over-estimate the level of the randomisation test. The bias may be unimportant when the number of missing observations is relatively small. Further, if we can regard the missing plot as one chosen at random from all the plots, the bias vanishes in both the cases considered in this paper.

(K. Rajaraman)

MODE, C. J. & ROBINSON, H. F. (Montana State Coll., Bozeman and N. Carolina State Coll., Greenboro) **7.2** (6.2)
Pleiotropism and the genetic variance and covariance—*In English*
Biometrics (1959) **15**, 518-537 (5 refs., 3 tables)

This paper extends the concepts of additive and dominance components of the genetic variance to genetic covariance. Genetic, genotypic, and phenotypic coefficients of correlation are defined and a method of estimation is given under the assumption that a gene affects unrelated characters; that is to say, its action is pleiotropic. To estimate the above coefficients of correlation and the average degree of dominance, it is necessary to perform an experiment where a random sample of males is mated to a random sample of females, each female is used once and only once.

Since the usual arguments would not hold in obtaining genetic covariance between different characteristics, the authors derived the variances and covariances of these means by direct calculations. Explicit formulae are given in terms of expectations and in terms of genetic components for all variances and covariances.

Design I of Comstock and Robinson [*Biometrics* (1948) **4**, 254-265] is reviewed and all formulae required to obtain the analysis of variance are given. The expectations of the several mean squares are presented and estimates of the various correlations are given under the absence and presence of epistasis. Variances of the correlations are obtained so that the reliability of the estimates may be obtained.

A numerical example is provided to illustrate the various computations that are involved.

(C. Y. Kramer)

ROBERTSON, A. (University of Edinburgh) **7.2** (4.1)
The sampling variance of the genetic correlation coefficient—*In English*
Biometrics (1959) **15**, 469-485 (4 refs., 4 tables, 1 fig.)

Genetic correlations may be estimated by (*a*) analysis of parent-offspring correlations or (*b*) analysis of variance and covariance components within and between relatives. Formulas for the sampling variance have been derived for (*a*), but not for (*b*). The author presents formulas for the sampling variance for (*b*), taking the special case in which heritabilities are equal for a given pair of characters.

Both variates are transformed into standard measure, then cast into a single analysis of variance. The between-group variance is partitioned into a mean square for groups and one for group-by-character interaction. Similarly, the within-group variance is divided into mean squares for individuals and for error. Variance of the genetic correlation coefficient is derived as a function of these four mean squares. The resulting formula then is converted to one which employs group size, number of groups, the intra-class correlation, and the within- and between-group correlations.

If the two variates under consideration are measured in different individuals, the estimated variance may be biased upward or downward, depending on group size and the magnitudes of the within- and between-group correlations. If both group size and the intra-class correlation are large, the variance of the genetic correlation approaches that of the correlation coefficient itself. If the genetic correlation approaches zero, its sampling variance increases, but can only be doubled in the most extreme cases.

The possibilities of a general solution, for cases in which the two characters have different heritabilities, are discussed. It is presumed that the appropriate formula will contain some joint function of the two heritabilities in its denominator.

A special use of the genetic correlation coefficient is to detect genotype-environment interactions. If the genetic correlation between performances in two different environments is less than unity, and the respective between-group variance components are not different, an interaction may exist. The biological importance of this interaction depends on (i) the deviation of the genetic correlation from unity and (ii) its standard error. The estimation of such standard errors is discussed, both within populations and over wider genetic bases. The case of more than two environments is also considered. In general, optimum group size in each environment is more than twice as large as for estimation of heritability.

Optimal structures of experiments for estimation of both heritabilities and genetic correlations appear to be the same. However, the sampling variance of the genetic correlation coefficient will be much larger.

The author discusses limitations of the foregoing approach. Primarily, the assumption that the two characters have the same intra-class correlation. Secondly that the genetic correlation coefficient is a ratio, and therefore its variance should be estimated from large samples.

(N. R. Thompson)

SRIVASTAVA, A. B. L. (Indian Institute of Technology, Kharagpur) **7.5** (3.2)

Effect of non-normality on the power of the analysis of variance test—*In English* *Biometrika* (1959) **46**, 114-122 (8 refs., 3 tables, 1 fig.)

The author discusses the power of the F-test for the one-way classification set-up when there are k groups of n observations. The parent population is assumed to have a distribution which can be represented by the first four terms of an Edgeworth series. The values of the parameters of this series are restricted so that the distribution has a functional form which is positive, definite, and unimodal. The method of attack on the problems is that which has been made familiar in a series of papers by Gayen [see, for example, *Biometrika* (1949) **36**, 353-369 and *Biometrika* (1950) **37**, 236-255].

The author finds that skewness of moderate order has little or no effect on the power of the analysis of variance test for the one-way classification considered. The effect of kurtosis on the other hand, is marked, especially for small samples. When the population is leptokurtic the effect is to increase the power near the significance level point and to cause it to decrease, by comparison with normal theory, in the region where the normal theory power is high. The reverse is true for platykurtois. Any effect appears to decrease in importance with increasing sample size. The general conclusion is that the effect of non-normality on the power function is of little consequence if the parent population is not widely divergent from that of normal.

(Florence N. David)

453

STEEL, R. G. D. (University of Wisconsin, Madison) **7.8** (5.6)

A multiple comparison rank sum test : treatments versus control—*In English* *Biometrics* (1959) **15**, 560-572 (5 refs., 3 tables)

In this paper the author presents a non-parametric test which is an analogue of Dunnett's procedure [*J. Amer. Statist. Ass.* (1955) **50**, 1096-1121] and also a generalisation of Wilcoxon's test [*Biometrics* (1945) **1**, 80-83]. The non-parametric nature of the test enables one to by-pass the usual analysis of variance assumptions. The test itself is a rank-sum multiple comparison test applicable in situations where there is a control and a number of treatments in a one-way classification; the number of observations on the control and each treatment is the same for all groups.

The proposed test is on the null hypothesis that the control and treatment variables have identical cumulative distribution functions. This hypothesis is tested against one of the following alternatives:

(i) At least one treatment variate is stochastically smaller than that of the control;
(ii) At least one treatment variate is stochastically larger than that of the control;
(iii) At least one treatment variate is stochastically different from that of the control.

All the cumulative distribution functions are by assumption continuous.

The measure for a single comparison is the minimum of the rank-sums on the control and a particular treatment when the two sets of values are ranked together in a single array. The proposed test statistic is then the minimum over all treatments of these minimum rank-sums. For a given experiment its empirical value is compared with the tabulated value as determined by a preassigned significance level α, the number of treatments k, and the number of observations on each treatment n.

In Section 4, the author discusses the exact distribution of $\min\{\min(T_i, T_i')\}$, where T_i and T_i' denote the rank-sums on a control and the ith treatment variables. This is followed by a table with exact probabilities for $k = 2$ and 3 and $n = 3, 4, 5$. In Section 5, it is assumed as an approximation that $(T_1, T_2, \ldots, T_k)$, where T_i is the sum of the ranks assigned to the control variable when ranked with the ith treatment, is from a multivariate normal distribution and that $[\min T_i - E(T_i)]/[\text{var}\,(T_i)]^{\frac{1}{2}}$ is distributed approximately as Dunnet's t with infinite degrees of freedom. Under the above assumptions, the author obtains approximate significant values of rank sums for both one-sided and two-sided alternatives at confidence level $1-\alpha = 0{\cdot}95$ and $0{\cdot}99$. The tabulated values are for cases where the number of treatments (excluding control) is $k = 2(1)9$ and the number of observations on each treatment is $n = 4(1)20$.

Section 3 is devoted to an example based on data obtained from the University of Wisconsin School of Medicine. Here, the author applies his work in testing which of three treatments results in lowering IQ scores below normal, and then demonstrates that application of Dunnet's procedure leads to the same conclusion.

(B. J. Trawinski)

454

van ELTEREN, PH. & NOETHER, G. E. (Math. Centre, Amsterdam and Boston Univ.) **7.7** (5.1)
The asymptotic efficiency of the χ_r^2-test for a balanced incomplete block design—*In English* *Biometrika* (1959) **46**, 475-477 (8 refs.)

This paper is concerned with the asymptotic relative efficiency in the sense of Pitman [see for example, Hannan, *J. R. Statist. Soc.* B (1956) **18**, 227-233] if the number of blocks tends to infinity of a distribution-free test for a balanced incomplete block design due to Durbin [*Brit. J. Psychol.* (1951) **4**, 85-90] with respect to the corresponding analysis of variance test. It is assumed that the underlying variables have distributions that may differ only in location parameters depending on block and treatment number. If these distributions are normal, the asymptotic efficiency $E = 3k/\pi(k+1)$ and in general E depends only on the block size k and not on the number of treatments n.

The asymptotic efficiencies of the following tests are obtained as special cases

1. $k = n$. Friedman's χ_r^2-test for designs with two classifications, also known as the method of m rankings.
2. $k = 2$. A test for the problem of paired comparisons asymptotically equivalent to a test of Bradley & Terry, discussed by Bradley [*Biometrika* (1955) **42**, 450-470].
3. $k = n = 2$. The sign test.

Earlier results of Bradley in the reference quoted above for case 2 and of Hodges & Lehmann [*Ann. Math. Statist.* (1956) **27**, 324-335] for case 3 do not quite agree with the results obtained in this paper, but it is shown that the differences are due to modifications in the methods of comparison.

(Ph. van Elteren)

WARTMANN, R. (Verein Dtsch. Eisenhüttenleute, Dusseldorf) **7.0** (2.2)
Computation with rounded or transformed values, especially in analysis of variance—*In German* *Biom. Zeit.* (1959) **1**, 190-202

This paper contains rules for fixing the accuracy of measurement or the rounding-interval in those cases where the variance of observed values is to be computed or where the values are to be subjected to an analysis of variance. The application of these rules leads to a considerable reduction of costs involved in the observation of the values and their evaluation.

The exact values x may be distributed around the expected value μ with the variance σ^2. If x^* stands for the rounded value corresponding to x, and c for the greatest possible difference between two corresponding rounded values, the rounding error $y = x^* - x$ lies in the interval $-c/2 \leqq y \leqq +c/2$. On the assumption that y is stochastically independent of x and that it is uniformly distributed in this interval, the expected values of x and x^* are equal. The variance of the rounded values is given by $\sigma^{*2} = \sigma^2 + c^2/12$. Accordingly, the relative increase of the variance due to rounding is: $U = (\sigma^{*2} - \sigma^2)/\sigma^2 = 1/12(c/\sigma)^2$. If a specified enlargement U is admissible, the rule $c = \sigma\sqrt{12U}$ ensues for c. If, for instance, $U = 1$ per cent., then $c = 0{\cdot}35\ \sigma$. Assuming that the values of measurement lie in a range of 5σ, it is sufficient to distinguish approximately $5 : 0{\cdot}35 = 14$ classes of values in the computation.

In computing the variance from many values it is common practice to classify the values according to approximately 10 to 20 classes. According to the above considerations, this simplification can also be utilised in those cases where only a few values are available. It is suggested that the values should be coded, say, between -10 and $+10$ and to round the transformed values off to integers. With the help of the values coded in this way it is easy to compute the mean value and the variance; the variance thus computed exceeds the variance of the original values only by the given proportion U.

This procedure can also be employed in the analysis of variance. Let σ_t^2 denote the total variance and U for the admissible proportion by which σ_t^2 may be enlarged by rounding. If $G+1$ specifies the order of magnitude of the variance-ratios F, the following relation holds for a 2-way variance-analysis with interactions: $c > \sigma_t\sqrt{12U}/\sqrt{1+2G/n}$, where n denotes the number of members in each cell. In an ν-way variance-analysis, in which interactions are taken into account only up to the second degree, the following approximation applies: $c > \sigma_t\sqrt{12U}/\sqrt{1+4G/n}$. On the assumption that $U = 1$ per cent. and $G = 7$, it is sufficient to use two digit integers lying in an interval of length 60. It is suggested that the original values are coded to integers which lie in the range ± 30.

By using the example of a 3-way variance-analysis it is shown numerically that the differences between the F-ratios of the exact values and of the coded values are relatively small. The slight extra work involved in the coding is largely compensated by the advantage of having convenient figures to work with in the analysis.

(K. A. Schäffer)

AROIAN, L. A. (Hughes Aircraft, Culver City, California) **8.9** (4.1)
Quality control charts by gauging—*In English*
Nat. Conv. Trans., Amer. Soc. Qual. Contr. (1959) 145-153 (4 refs., 4 tables, 2 figs.)

It is assumed that a characteristic, x, of an item is normally distributed. Where exact measurements of x are expensive, it is suggested that gauging be used. The first gauge will be used to classify all items less than x_1 and a second gauge to classify all items above x_2. Let p, q and r represent the true proportion of the population below x_1, between or equal to x_1 and x_2, and above x_2, respectively.

In a sample of n items, let a represent the number of items below x_1 and c represent the number of items above x_2. Thus $(c-a)$ is an indicator for a shift in the mean and $(c+a)$ is a measure of variability. Control charts can be constructed based on these two statistics.

Since the underlying distribution is normal, $p = r$ is chosen for control of the mean. Tables for $(c-a)$ are given where p ranges from 0·05 to 0·40 for samples of size $n = 8$, 10 and 12. Corresponding to each value of p, for a particular n, the control limits and the probability, 2α, that $(c-a)$ will exceed the control limits when the process is in control are given. Control limits for values of 2α between 0·002 and 0·05 are listed. For each combination of p, n, and 2α, the sample size required by using the ordinary $\bar{x}$ chart is given. The sample size was determined such that approximately the same operating characteristic curves were obtained for the $(c-a)$ chart and the $\bar{x}$ chart. The comparison is made for the $\bar{x}$ chart where the estimate of the standard deviation is obtained from the range, R. Efficiencies up to approximately 75 per cent. of the $(c-a)$ chart compared to the $\bar{x}$ chart were obtained.

A similar study of the $c+a$ chart for variability is given. Efficiencies from 70-80 per cent. of the $c+a$ chart compared to the R chart are obtainable.

Since the $(c-a)$ and $c+a$ charts are not independent, it is suggested that they be used in conjunction corresponding to the joint use of $\bar{x}$ and R charts.

Estimates of the mean and standard deviation from the gauged data are

$$\bar{x}' = (x_1+x_2)/2-(u_1+u_2)\sigma'/2$$
$$\sigma' = (x_2-x_1)/(u_2-u_1)$$

where $\Pr(u \leqq u_1) = a/n$ and $\Pr(u \geqq u_2) = c/n$, and u is the standardised normal deviate.

(D. W. Gaylor)

DAVID, H. T., FAY, E. A. & WALSH, J. E. (Iowa State Coll., U.S. Naval Test Sta. and Lockheed Aircraft Corp.) **8.8** (8.7)
Acceptance inspection by variables when the measurements are subject to error—*In English*
Ann. Inst. Statist. Math., Tokyo (1950) **10**, 107-129 (3 refs., 1 table)

In this paper an acceptance inspection procedure is discussed where the observable variable is composed of two parts; one from the fluctuation of the characteristic (true value) of the sampled item and the other from the measurement error. The measurement is supposed to be performed m-times on a sampled item and the measured values are averaged to give the variable x_i for the inspection. The true values are supposed to follow a normal distribution with unknown mean μ and unknown variance σ^2. The measurement errors are supposed to be mutually independent and to follow one and the same normal distribution with zero mean and variance. It is assumed that the constants R and $\bar{R}$ which satisfy the relation $0 < R \leqq \sigma/\sigma_e \leqq \bar{R} \leqq +\infty$ are known.

The main concern of this paper is where σ_e is known and the acceptance criterion is represented as "Accept the lot if and only if $\bar{x}+k_s-v\sigma_e \leqq U$" for some fixed constants k, v and U where $\bar{x}$ and s^2 are the mean and variance of sample $(x_1, x_2, \ldots, x_N)$. It is further assumed that the distribution of s is approximated by the normal distribution with mean $(\sigma^2+\sigma_e^2/m)^{\frac{1}{2}}$ and variance $(\sigma^2+\sigma_e^2/m)/2N$. It is required that the acceptance criterion has the property that the probability of accepting the lot is greater than $1-\alpha$ when the probability for the true value to be greater than U is less than p_1, and the probability of rejecting the lot is greater than $1-\beta$ when the probability for the true value to be greater than U is more than p_2.

The paper proceeds with a statement of results for the case when the variance of the measurement errors is known and the case where it is not known. The properties of the upper limit $(\bar{R})$ of the ratio of the standard deviation of the measurement errors (σ/σ_e) are given together with the determinations of the number of measurements (m) to be made on each item of the sample. A brief note is given on the properties of a test proposed by Wallis [*Selected Techniques of Statistical Analysis* (1947), McGraw-Hill, chapter 1] together with a numerical example.

The paper concludes with a detailed section showing the various derivations used in the development of the paper.

(H. Akaike)

FRAZIER, D. & WHITMAN, D. R. (Standard Oil Co. Ohio and Case Inst. Tech., Cleveland) **8.9** (10.2)
Quality control applications in the oil refining field—*In English*
Nat. Conv. Trans., Amer. Soc. Qual. Contr. (1959) 201-212 (4 figs., 1 graph).

The usual techniques of quality control apply to the production of discrete things. In the petroleum industry a continuously flowing material is produced. There is no natural unit of production and mixing can be used to average out fluctuations in the material. Thus, modifications of conventional quality control techniques are required, since the assumption of independence from one sample to the next is not valid.

A sample is contaminated to some extent with the material present when the previous sample was taken. The model considered is

$$(P_i - \rho P_{i-1})/(1-\rho) = A + E_i,$$

where P_i is the result obtained from the ith sample, P_{i-1} the result from the previous sample, ρ is the proportion of the material still present from the time of the previous sample, A is the average or target value, and the E_i's are independent errors from a normal population with zero mean. The assumption that the E_i's are independent is justified if the fluctuations about A occur rapidly compared to the length of time between samples. If samples are taken at regular intervals of Δ, then $\rho = \exp(-R\Delta/V)$ where R is the flow rate and V is the effective volume of the mixing tank. It is suggested that R be set so that $0{\cdot}1 < \rho < 0{\cdot}9$. Since the quantity $(P_i - \rho P_{i-1})/(1-\rho)$ varies independently about the target value A, conventional quality control techniques can be applied.

The final product is collected and sold in batches. As the batch size decreases, the batch rejection rate will increase. As the batch size increases, there is a greater chance for process fluctuations to average out. A graph is given from which the standard deviation among batches can be found from the variation among stream samples. Then, the average or target value can be set depending on the probability required that the batch passes some specification.

It is desirable to estimate the final batch quality from samples taken from the stream during production. Graphs are given from which the standard deviation among samples with a batch can be found. From the graphs, the optimum number of samples could be determined in order to insure that the batch has a given probability of satisfying specifications.

(D. W. Gaylor)

459

MICKEY, M. R. (General Analysis Corporation, Los Angeles) **8.2** (8.1)
Some finite population unbiased ratio and regression estimators—*In English*
J. Amer. Statist. Ass. (1959) **54**, 594-612 (8 refs., 2 tables)

In this paper, a broad class of unbiased ratio and regression-type estimators is given which includes the well-known unbiased ratio estimator of Hartley and Ross. The idea utilised in the construction of such estimators is used to construct estimators of the population total when sampling is done with varying probabilities of selection.

In Section 2, a broad class of unbiased ratio and regression-type of estimators is constructed, and in Section 3 examples of some useful unbiased ratio and regression-type estimators are given which includes Hartley and Ross estimator. The variance of these estimators is not considered, but in Section 4 unbiased non-negative estimators of variance for a subclass are constructed, and Section 5 gives numerical examples.

In Section 6, a set of unbiased estimators with non-negative estimators of variance is constructed, when sampling is done with varying probabilities of selection. This overcomes the defect of Yates and Grundy's estimator of variance which can take negative values.

(J. N. K. Rao)

MILNE, A. (King's College, Newcastle-upon-Tyne, England) **8.6** (8.4)
The centric systematic area-sample treated as a random sample—*In English*
Biometrics (1959) **15**, 270-297 (26 refs., 9 tables, 6 figs.)

It is well known that systematic samples may produce biased estimates of the mean and variance if periodic variations occur in the population. On the other hand, if the population is randomly ordered, systematic samples may be treated as random samples without incurring bias. However, little is usually known about possible periodic variations in actually sampled populations, and hence existence of bias can only be hypothesised.

The author has made an attempt to obtain some estimates of bias due to the use of the centric systematic sample on forty-nine biological and one artificial population on which complete enumeration data were available. The centric systematic sample is drawn, from a population distributed on a plane, such that sample points lie on equi-distant parallel lines, and is thus well suited for sampling certain biological populations.

The author uses each of these populations to draw an equivalent simple random sample, stratified random sample, and centric systematic sample, the latter being analysed as (i) a simple random sample and (ii) a stratified random sample. The resulting estimates are then compared, particularly with reference to the estimates obtained through the centric systematic samples.

In general, the author finds that the estimates from the centric systematic samples compare favourably with these from the other samples. The existence of hypothetical periodic variations in the population is discussed, particularly for one of the populations where the centric systematic sample fared relatively poorly. In this case he strongly suspects that the periodic variation that caused the bias in the centric systematic sample is due to an enumerator bias. Thus the author's final conclusion is that, "with intelligent caution, one will not go very far wrong, if wrong at all, in using CSS (centric systematic sample) as if it were random".

(R. J. Freund)

NISSELSON, H. & WOOLSEY, T. D. (Bureau of Census and Public Health Service, Washington) **8.5** (7.2)
Some problems of household interview design for the National Health Survey—*In English*
J. Amer. Statist. Ass. (1959) **54**, 69-87 (21 refs., 9 tables)

Evidence from earlier surveys and from a pilot survey conducted in Charlotte, North Carolina, provided some answers in making decisions about the conduct of the interview for the National Health Survey.

Data on illness conditions from households, in which all adults were to be interviewed for themselves, were compared with those obtained from households in which adults found at home were to be interviewed for themselves and absent related household members. The sampling and response variability are too high to permit definite conclusions as to possible biases in the use of a household respondent. From cost considerations and experience from previous surveys the following procedure was adopted: information for an adult at home at the time of the interview is to be obtained from the person himself; for an absent adult only from the person's spouse, if he is married, or from a parent or adult son or daughter residing in the household. Information for adults not related to the head of the household is to be obtained from the person himself, or from a proxy respondent related to him as above, if one is living in the household.

Particular attention was given the problem of variability and bias which may be associated with the interviewer. Estimates were constructed for the variance between interviewers and the variance within interviewers for the statistic "rate per person" based on all persons. The effects are more marked when all adults are interviewed for themselves, and more marked for males interviewed for themselves than for females interviewed for themselves. Since all interviewers were female, the difference in sex between the interviewer and respondent may itself be a source of non-sampling variance.

The use of check lists of diseases resulted in the reporting of considerable numbers of medically attended and a few currently disabling chronic conditions. The question of "optimum" length of such check lists arises. Another related problem is the order in which the items were read during the survey. In the Baltimore Health Survey, definite order effects were found for the symptoms, but the sample was not large enough to detect order effects in the chronic condition list. The use of probe questions may be a superior method of eliciting reports of chronic disease.

Another problem considered is the recall of illness events in a specified period of time. It was found that the recall was better for acute illnesses than episodes of illness caused by a chronic condition, and that it was better for disabling and for medically attended acute illnesses than for acute illnesses involving neither disability nor medical care. As a result of the Charlotte pilot survey, two weeks was adopted as the recall period for all illness, injuries, and medical and dental care visits, except that for hospital illness a recall period of a year was maintained.

(B. T. Oñate)

ODERFELD, J. (Math. Inst., Polish Academy of Sciences, Warsaw) **8.8** (6.4)

Statistical testing for correlated characteristics—*In Polish*

Zastosowania Mat. (1959) **4**, 255-264 (2 refs., 1 table, 2 nomograms)

Let the two dimensional random variable (X, Y) have a distribution with known mean deviations, a known correlation coefficient and unknown means μ_X and μ_Y.

Let X be a characteristic that is easy to test but not very important, and Y a characteristic that is difficult to test but decisive as regards the usefulness of the subject-matter. We can then test the characteristic X in a large sample, or even in a very large sample, i.e. the whole lot, obtaining an almost exact estimate of μ_X. The characteristic Y, however, can be tested only in a small sample.

Combining these data in a suitable way and assuming that the distribution of (X, Y) is a two-dimensional normal distribution, the author describes an acceptance test which will verify statistical hypotheses concerning the fraction defective of the lot. The power function are given and it is shown that the test suggested is in certain conditions more economical than the classical one.

Subsequentially the author constructs another test. In this test the sample is ordered with respect to characteristic X and only one item, having the mth extreme value of X in the sample ($m = 1, 2, 3$), is tested for Y.

(J. Oderfeld)

ROBERTS, S. W. (Bell Telephone Labs., New York City) **8.9** (5.2)

Early detection of relatively small shifts in a process average—*In English*

Nat. Conv. Trans., Amer. Soc. Qual. Contr. (1959) 73-78 (10 refs., 8 figs.)

Points which are averages of n items each are plotted on control charts based on a process average of μ_0 and a standard deviation of σ'. The effectiveness of the control chart is summarised by a plot of $\mathscr{E}(y)$ versus Δ, where $\mathscr{E}(y)$ is the expected number of points occurring before a shift to $\mu_0+\delta$ is detected and $\Delta = \sqrt{n}\,\delta/\sigma'$. Only one-sided limits are considered but an approximate method is given for obtaining curves for two-sided limits. Curves of $\mathscr{E}(y)$ versus Δ are plotted for seven procedures for rejecting the hypothesis, $\delta = 0$.

(i) Reject when one point falls above $\mu_0+L\sigma'/\sqrt{n}$. Curves are presented for $L = 2{\cdot}0(0{\cdot}5)3{\cdot}5$.

(ii) Reject when k consecutive points fall above $\mu_0+L_k\sigma'/\sqrt{n}$. Curves are presented for $k = 1(1)4$, 6, 8 where the L_k are chosen so that all tests have the same $\mathscr{E}(y)$ at $\Delta = 0$. For small Δ, a large value of k is best and vice versa.

(iii) Reject if one point falls above $\mu_0+3\sigma'/\sqrt{n}$ or seven consecutive points fall above μ_0.

(iv) Reject if the moving average of k consecutive points exceeds $\mu_0+3\sigma'/\sqrt{(nk)}$. Curves are presented for $k = 1(1)5$, 8.

(v) Reject if z_j exceeds $\mu_0+3\sigma'\sqrt{\{r/n(2-r)\}}$, where z_j is a geometric moving average with $z_0 = 0$ and $z_j = (1-r)z_{j-1}+r\bar{x}_j$, $j>0$, $0<r\leqq 1$, where r is arbitrary. The means, $\bar{x}_j$, are assigned geometrically decreasing weights back to the first mean. Curves are given for $r = 0{\cdot}25$, $0{\cdot}5$, $1{\cdot}0$.

(vi) Reject when a point exceeds $\mu_0+3\sigma'/\sqrt{n}$ or when the moving average of k points exceeds $\mu_0+3\sigma'/\sqrt{(nk)}$. Curves are given for $k = 1$, 2, 4, 8.

(vii) Reject when a point exceeds $\mu_0+3\sigma'/\sqrt{n}$ or z_j exceeds the limit given in V.

For shifts greater than $2\sigma'/\sqrt{n}$ the standard control chart is most practical. For smaller shifts, some of the other types may be useful.

(D. W. Gaylor)

SAVAGE, I. R. (University of Minnesota, Minneapolis) **8.8** (1.8)
A production model and continuous sampling plan—*In English*
J. Amer. Statist. Ass, (1959) **54**, 231-247 (24 refs., 3 tables)

The author considers a production process subject to random incidence of flaws. The cumulative number of flaws $z(t)$ at time t since the last trouble-shooting and overhaul is assumed to be a Poisson process; the probability of producing an effective item at time t is assumed to be the decreasing function $PR^{z(t)}$ of $z(t)$, $0<P$, $R<1$.

Let F be the expected time between inspections; define cycle length as the (random) time between successive trouble-shootings, trouble-shooting being initiated as soon as a defective item is found. Then profit, defined as the expected value of income per cycle divided by the expected value of cycle length, is a function of F and several economic parameters. An interesting aspect of these economic parameters is that both the duration and the cost of trouble-shooting and overhaul is made to depend on the number of flaws z that have accumulated since the last overhaul.

The author shows, by example and also in general, how to compute the profit-maximising F, given specified economic parameters.

(H. T. David)

TRYBULA, S. (Math. Inst., Polish Academy of Sciences, Wroclaw) **8.8** (4.0)
Estimation taking into account the error of the controller—*In Polish*
Zastosowania Mat. (1959) **4**, 249-254 (3 refs., 1 table)

The quality controller makes errors in the course of inspecting items of a sample. Let us denote by p_1 and p_2 the probability of recognising a good item as a bad one and a bad item as a good one. The author considers the problem of estimating the percentage defective in the lot from which the sample of size n has been taken (with replacement). The most effective unbiased estimator, i.e.

$$\{1/(1-p_1-p_2)\}\{m/n-p_1\},$$

where m is the number of items qualified as bad, is not suitable. This is mainly because for some values of m it assumes negative values or values greater than unity. The author proposes using another estimator, which is a Bayes solution under the assumption that the prior distribution of percentage defective is uniform.

A numerical table for $n = 20$ illustrates the divergence between the two estimators.

(S. Zubrzycki)

VAGHOLKAR, M. K. (Bombay Textile Research Association) **8.8** (8.7)

The process curve and the equivalent mixed binomial with two components—*In English*
J. R. Statist. Soc. B (1959) **21**, 63-66 (4 refs.)

The author is concerned with the form of the process curve assumed for the mathematical analysis of inspection plans on a cost basis. Barnard [" Sampling inspection and statistical decisions," *J. R. Statist. Soc.* B (1954) **16**, 151-174] proposed using a two point distribution and this paper sets out in detail a method whereby this may be achieved. Batches of articles are inspected for acceptance, p being the proportion defective in the batch. If $p > p_0$ the batch is " bad " ; if $p < p_0$ the batch is acceptable. $L(p)$ is the decision loss from a wrong decision [$L(p_0) = 0$], and $P(p)$ is the prior distribution of p among batches presented for inspection, or $P(p)$ is the process curve. The total expected cost per batch inspected is to be a minimum.

The author takes

$$P\{p = p_1\} = a_1,\ P\{p = p_2\} = a_2 = 1 - a_1,$$

a_1 is shown as equal to the integral of $P(p)$ with respect to p from zero to p_0. p_1 is chosen so that $L(p_1)$ is equal to the integral of the product $P(p)L(p)$ with respect to p from zero to p_0 divided by a_1. p_2 is chosen so that $L(p_2)$ is equal to the integral of the product $P(p)L(p)$ with respect to p from p_0 to 1, divided by a_2.

Examples are given and the procedure compared with those of Champernowne and Sittig.

(Florence N. David)

467

VEEN, B. (N.V. Onderzoekingsinstituut " Research ", AKU, Arnhem) **8.1** (8.7)

Optimum allocation in nested sampling—*In Dutch*
Statist. Neerlandica (1959) **13**, 171-186

The solution to the problem of optimum allocation which is given by Marcuse [*Biometrics* (1949) **5**, 189] is explained and elucidated by an example. The known formulae obtained via this method often do not sufficiently approximate the true (integer) optimum sample numbers. This paper shows how this can be avoided.

If one has to deal with a design for a nested sampling procedure, the formulae given by Marcuse make it possible to calculate the number of observations within each stage of the nested design. Some conditions, however, have to be fulfilled.

(i) A reliable estimation of the components of variance within each stage must be known,
(ii) the costs of observation per unit within each stage must be known.
(iii) the allowed total costs or the allowed variance of the sample mean must be specified.

In such a case it is possible to minimise with respect to the variance of the sample mean, or the necessary costs by choosing suitable numbers of observations within each stage of the sampling design.

In practice it turns out that considerable savings can be realised by this method. However, it appears that if small sample numbers, unfavourable ratio of costs and variance, and designs with more than two stages are concerned, the known formulae often do not sufficiently approximate the true optimum sample numbers. The departures are caused by the discrete character of the problem. Strictly speaking, the formulae only hold for continuous cases. In practice, however, it is only possible to take an integer number of samples. This requires rounding off the fractional numbers found by means of the formulae. But, by doing this, we leave the optimum allocation of the sample numbers and the problem is *how* to round off to get the set of sample numbers which give the best solution. It appeared that generally the true optimum cannot be reached by applying simple rules for rounding off to the nearest integers. A number of possible ways for finding the required optimum sample numbers are described. These are based on calculation of the results for different combinations of integer numbers in the " neighbourhood " of the fractional numbers calculated from the formulae.

In an appendix a short theoretical consideration is given about the optimum relation between variances (V) and costs (C), especially with respect to the integer character of the sample numbers. Each set of sample numbers gives one point (V, C). An optimum point (V_0, C_0) is defined as a point characterised in that there is no other point (V, C) of which both co-ordinates are at the same time equal to or smaller than those of the point (V_0, C_0) under consideration. With the aid of this definition it is possible to construct an " optimum path " of points, which can be used to allocate the required optimum in a certain case.

(B. Veen)

468

ANDERSON, R. L. (North Carolina State College, Raleigh) **9.6** (6.9)
Use of contingency tables in the analysis of consumer preference studies—*In English*
Biometrics (1959) **15**, 582-590 (4 refs., 3 tables)

Consider a preference study in which n consumers are asked to rank r varieties. In the resulting data it is desired to test for independence of varieties and ranks. This is not the usual problem of a contingency table with fixed border totals since, although each of the n consumers act independently, he has only $r!$ possible preference sequences. Every consumer is represented in each column (rank) so that one may make a test of varietal differences at each rank, using a χ^2 statistic (denoted by Q_j^2 for rank j; $j = 1, \ldots, r$) based on $r-1$ degrees of freedom. However, since the Q_j^2, $j = 1, \ldots, r$ are not independent, $Q^2 = \sum_{j=1}^{r} Q_j^2$ does not approach a χ^2 with $r(r-1)$ degrees of freedom. For an over-all test the statistic $(r-1)Q^2/r$ is proposed.

It is shown that the proposed test statistic has an asymptotic χ^2-distribution with $(r-1)^2$ degrees of freedom. Reference is made to a sampling study of the distribution of Q^2 for $r = 3$ with $n = 30$.

Four alternative methods of analysing consumer preference data of this kind are considered. The first is a test of independence of the first and second of three rankings. The second involves construction of orthogonal single-degree-of-freedom contrasts. The third alternative makes use of a likelihood procedure for the rank analysis of triple comparisons, while the fourth involves analysis of the variation among scores assigned to the varieties.

(W. A. Glenn)

BOSE, R. C. & CARTER, R. L. (Univ. North Carolina, Chapel Hill, and Illinois Inst. Tech., Chicago) **9.3** (9.2)
Complex representation in the construction of rotatable designs—*In English*
Ann. Math. Statist. (1959) **30**, 771-780 (4 refs.)

Response surface techniques are discussed as a generalisation of factorial designs, emphasising the concept of rotatability; the property that the information (reciprocal of the variance) contours are hyper-spheres centered at the origin of the k-dimensional sample space where k denotes the number of factors. For the case of two factors, it is shown that the necessary and sufficient condition for a configuration of sample points to be a rotatable arrangement of a specified order d, is greatly simplified if the factor space is considered as the complex plane. In fact the condition is that

$$\sum_u z_u^{a-b} \bar{z}_u^b = 0$$

for all integers a and b satisfying $0 < a \leqq 2d$, $0 \leqq b < a/2$ where z_u is the complex variable $x_u + iy_u$, $\bar{z}_u$ is its complex conjugate $x_u - iy_u$ and (x_u, y_u) is the uth sample point. Necessary conditions governing the combination of rotatable arrangements into configurations possessing a higher order of rotatability are discussed.

These conditions are also sufficient for circular rotatable arrangements where a rotatable arrangement formed by N points $z_1, z_2, \ldots, z_N$ of the complex factor plane, is called a circular rotatable arrangement if the N points lie on a circle with the origin as centre. This is done by showing that certain coefficients must vanish in the " design equation " whose roots are the (complex) values of the various sample points. A method is presented by which any configuration of sample points (for example, some configuration fixed by non-statistical conditions) may be completed into a rotatable design of the first order by the addition of only two properly chosen additional sample points.

The authors mention that one of them has, by using an iterative process, calculated a table of circular rotatable arrangements of the first order each with seven points not situated at the vertices of a regular heptagon, and that it is hoped to publish the details of the computational procedure and the table of designs as a separate paper.

(J. W. Wilkinson)

BOSE, R. C. & DRAPER, N. R. (University of North Carolina, Chapel Hill) **9.3** (9.2)
Second order rotatable designs in three dimensions—*In English*
Ann. Math. Statist. (1959) **30**, 1097-1112 (9 refs., 3 tables, 2 figs.)

This paper is devoted entirely to the construction of second order rotatable designs in three dimensions, i.e. for three factors. By the use of new construction methods various infinite classes of designs have been obtained. These infinite classes can be classified into three main types according to their method of construction. Ten infinite classes of designs of one type are tabled with the number of points in each design ranging from 18 to 32. Four infinite classes of designs of another type are tabled with the minimum number of points in each design ranging from 29 to 32. The third type contains one infinite class of 16 point designs. A table giving a partial listing of these designs is presented.

Certain particular designs can be generated by the methods of construction that are used. Only two particular designs contain fewer points than the infinite class of 16 point designs, these having 12 and 14 points, respectively. The authors point out that it may be shown that all the second order rotatable designs in three factors previously known can be derived as special cases of these infinite classes.

(J. W. Wilkinson)

471

CHERNOFF, H. (Stanford University, California) **9.7** (5.7)
Sequential design of experiments—*In English*
Ann. Math. Statist. (1959) **30**, 755-770 (6 refs., 2 tables, 1 fig.)

This paper presents a procedure for the sequential design of experiments where the problem is one of testing a hypothesis H_0. It is assumed there are two possible actions, viz., accept H_0 or reject H_0, and a class of available experiments. After each observation the statistician has to take the decision whether to seek further observations. If he decides to continue, he must select one of the available experiments. If he decides to stop he must select one of the two terminal actions.

The approach used is motivated by Wald's sequential probability ratio test for a simple hypothesis compared with a simple alternative when only one experiment is available. The risks corresponding to such a strategy are known to be functions of the probabilities of incorrect decision, cost of experimentation and the average number of observations. The average number of observations is a function of the Kullback-Liebler information numbers given by

$$I_0 = \int \log [f_0(X)/f_1(X)] f_0(X) dx,$$

$$I_1 = \int \log [f_1(X)/f_0(X)] f_1(X) dx,$$

f_0 and f_1 being the densities of the observed random variable under the hypothesis and alternative respectively.

In the more general case where there are several experiments available at each observation time, and where the hypotheses are composite, the choice of experiment at each stage is based on these information numbers and the maximum likelihood estimate $\hat{\theta}_n$ of θ. If $\hat{\theta}_n$ is consistent with H_0, then that experiment E_0 is chosen which maximises $I_0(E)$. If $\hat{\theta}_n$ is consistent with H_1, then the experiment E_1 chosen is that one which maximises $I_1(E)$. The stopping rule used is analogous to that of the usual sequential probability ratio test.

This procedure is shown to be " asymptotically optimal " as the cost of sampling tends to zero, which in turn implies that sample sizes will be large. As the author points out, the asymptotic optimality may not be relevant in the early stages of experimentation—especially if the cost of sampling is high. The procedure here is based essentially on assuming the maximum likelihood estimate at any stage is close to reality and it is efficient when this is so.

(D. G. Chapman)

472

GIRI, N. C. (Indian Council of Agric. Res., New Delhi) **9.1** (7.4)

On row balance in partially balanced incomplete block designs—*In English*

J. Indian Soc. Agric. Statist. (1957) **9**, 168-178 (6 refs.)

In this paper, it is shown that the following two classes c_1 and c_2 of partially balanced incomplete block designs with parameters:

$$c_1 : v = p(p-1), \quad b = p^2, \quad r = p,$$
$$k = p-1, \quad n_1 = p(p-2), \quad n_2 = p-1$$
$$\lambda_1 = 1 \quad \lambda_2 = 0 \quad p_{ij}^1 = \begin{bmatrix} p(p-3) & (p-1) \\ (p-1) & 0 \end{bmatrix}$$
$$p_{ij}^2 = \begin{bmatrix} p(p-2) & 0 \\ 0 & p-2 \end{bmatrix}$$

and

$$c_2 : v = pq, \quad b = p^2, \quad r = p, \quad k = q, \quad n_1 = p(q-1)$$
$$n_2 = p-1, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad p_{ij}^1 = \begin{bmatrix} p(q-2) & (p-1) \\ (p-1) & 0 \end{bmatrix}$$
$$p_{ij}^2 = \begin{bmatrix} p(q-1) & 0 \\ 0 & p-2 \end{bmatrix}$$

for $p = 4, \quad q = 2, 3;$
$p = 5, \quad q = 2, 3, 4;$
$p = 7, \quad q = 2, 3, 4, 6;$
$p = 8, \quad q = 2, 3, 4, 7;$
$p = 9, \quad q = 2, 3, 4, 5, 8$
and $p = 10, \quad q = 2, 3, 5, 9$

can be arranged in rows such that each treatment is replicated at least m times and at most $(m+1)$ times in each row and the arrangement of the $(m+1)$th replicate of the treatments in the rows is such that these form a partially balanced incomplete block design with parameters

$$v' = v, \quad b' = k, \quad r', k', \lambda_1', \lambda_2', \quad n_i' = n_i$$
$$p_{ij}^{t'} = p_{ij}^t \text{ for } i, j, \quad t = 1, 2.$$

The general intra- and inter-block analysis for these designs eliminating the row effects is also presented.

(J. Roy)

GRAYBILL, F. A. & WEEKS, D. L. (Oklahoma State University) **9.1** (7.3)

Combining inter-block and intra-block information in balanced incomplete blocks—*In English*

Ann. Math. Statist. (1959) **30**, 799-805 (5 refs., 1 table)

For balanced incomplete block designs associated with the Eisenhart Model III (blocks random, error random), two independent estimates of treatment differences, as well as a combined estimate, were given by Yates. This paper expresses the combined estimate in terms of a set of minimal sufficient statistics and shows that a slightly altered form of the estimate is unbiased.

The mathematical model is written as

$$y_{iqm} = \alpha + \beta_i + \tau_q^* + e_{iqm},$$

where $i = 1, 2, \ldots, b$; $q = 1, 2, \ldots, t$; and $m = 0, 1, \ldots, n_{iq}$. Here t is the number of treatments; b is the number of blocks; n_{iq} is 1 if the qth treatment is in block i and 0 elsewhere; and $n = bk = tr$, where k is the number of cells per block and r the number of replicates. A transformation is made so that the parameters to be estimated are σ^2 (the common variance of e_{iqm}), σ_β^2 (the variance of the β_i), $\mu = \alpha + t^{-1} \Sigma_q \tau_q^*$, and τ (a vector whose $t-1$ elements are contrasts of the τ_q^*). The set of minimal statistics is given at the end of Section 4 and consists of $(t-1)$ vectors $U = (u_i)$ and $X = (x_i)$ and scalars $y\ldots$, S^{*2}, and S^2. In Section 5 it is pointed out that u_i is an unbiased (intra-block) estimate of τ_i; x_i is an unbiased estimate (inter-block) of τ_i; S^2/f is an unbiased estimate of σ^2, where $f = n-b-t+1$; $S^{*2}/(b-t)$ is an unbiased estimate of $\sigma^2 + k\sigma_\beta^2$; and $y\ldots$ is an unbiased estimate of μ.

Yates's combined estimate of τ_i has two forms, one if $\hat{\sigma}_\beta^2 > 0$, another if $\hat{\sigma}_\beta^2 \leqq 0$. In Section 7, the first of these estimates is written in terms of U, X, S^{*2} and S^2. In Section 8 it is shown that if this first estimate is used, regardless of $\hat{\sigma}_\beta^2$, then it provides an unbiased estimate of the τ_i.

(R. Hooke)

GRIDGEMAN, N. T. (Div. of Applied Biology, Nat. Res. Council, Canada) **9.5** (9.6)

Pair comparisons, with and without ties—*In English*

Biometrics (1959) **15**, 382-388 (9 refs., 3 tables)

A probabilistic model for paired comparisons with the possible admission of ties is treated in this paper. The model's five parameters are the probabilities of the following events:

a. Discrimination of the difference between items A and B.
b. Deliberate choice of item A, given discrimination;
c. Deliberate choice of item B, given discrimination;
d. Arbitrary choice of A or B, given discrimination;
e. Indiscriminate guessing.

The probabilities of selecting A or B, or of reaching a tie, are expressed in terms of these parameters. Admitting ties, the probabilities associated with discrimination tests between two stimuli of different intensity are functions of the probabilities of events (*a*) and (*e*); excluding ties, only the former probability is a parameter. Preference trials with obviously different items and possible ties involve the probabilities of (*b*), (*c*) and (*d*), while the model without ties depends only upon the probabilities of (*b*) and (*c*).

It is argued that the admission of ties in paired comparisons (followed by their omission from the analysis) increases the power of the test of the null hypothesis that the probability of discrimination ((*a*) above) between items A and B is zero. However, exceptions may occur for small numbers of repetitions, and tables are given to illustrate this behaviour of the power function. An empirical study of the effect of admitting ties is presented with data from a taste-testing experiment of known properties, with the conclusion that the presence of ties depresses the estimate of the probability of (*a*).

(D. F. Morrison)

LIEBLEIN, J. (David Taylor Model Basin, Maryland) **9.2** (11.5)

A general analysis of variance scheme applicable to a computer with a very large memory—*In English*

J. Ass. Comp. Mach. (1959) **6**, 469-475 (1 ref.)

Assume that the observations obtained in an r-factor factorial experiment, in which the ith factor has n_i levels, have been stored in a computer in sequential order with the first number in cell 0000 and the last in cell $n_1n_2n_3\ldots n_r$. The complete analysis of variance is based on linear combinations of the squares of sums of the observations. This paper gives a method for computing these sums on an electronic computer.

Since the observations are in order, the sums for the first factor can be obtained by summing the first n_1 values for the first sum, the next n_2 values for the second sum and so on. Similar sums for the second factor could be obtained by rearranging the order so that the observations were in order by the second factor. In an electronic computer this procedure is wasteful of computing time and other methods may be more efficient.

The method developed by the author consists of computing the locations of all the observations which should be added together to obtain a certain sum. The basis result of this paper is a formula by which the locations of all the observations which should be added into a certain sum can be obtained. An example is given to illustrate the use of this formula.

(D. Teichroew)

HARTLEY, H. O. (Iowa State University, Ames) **9.3** (4.3)

Smallest composite designs for quadratic response surfaces—*In English*

Biometrics (1959) **15**, 611-624 (4 refs., 3 tables)

Composite designs were introduced by Box and Wilson [*J. R. Statist. Soc.* B (1951) **13**, 1]. These designs add certain additional points to the 2^n factorial design (sometimes called " star designs "). This paper gives a simple answer to the problem of determining how many and what coefficients of the quadratic response surface of the form

$$Y = \beta_0 + \sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n} \beta_{ii} x_i^2 + \sum_{i<j}^{n} \beta_{ij} x_i x_j$$

can be estimated. However, composite designs are examined only with regard to their merits as a single (one-piece) experiment which is supposed to supply directly estimates of the parameters in the above-mentioned response surface.

The designs derived are based upon the following theorem:

(*a*) In any composite design, in which no main effect is used as a defining contrast of the $(1/2^k)2^n$ fractional replicate, it is always possible to estimate the following coefficients of the quadratic response surface given above: all linear coefficients (b_i), all quadratic coefficients (b_{ii}), and the constant (b_0); one of the product coefficients (b_{ij}) selected from each of the alias-sets.

(*b*) It is not possible to estimate more than one of the product coefficients (b_{ij}) from each alias-set.

The implication of this theorem is that the largest number of coefficients of that response-surface can be estimated if each alias-set of the fractional replicate contains at least one two-factor interaction $x_i x_j$ permitting the selection of one coefficient b_{ij} from each of the alias-sets.

Examples of such designs are then discussed. The design matrix, **X**, for the general case is also given. In order to form $\mathbf{XX}'$ the coefficients b_0, b_i, b_{ii}, b_{ij} must be arranged into groups as follows:

A. The even order group: b_0, b_i; $i = 1, 2, \ldots, n$.

B. One alias group for any of those of the $2^n - 1$ alias-sets which contain either at least one main effect or have a two-factor interaction $x_i x_j$ represented in the p selected terms.

Examples of both groups are given in considerable detail.

The variances and covariances of the estimated coefficients are then considered. Since any two coefficients in two different alias groups are uncorrelated the author confines himself to a single alias group and again distinguishes the two cases:

(i) A product term $x_i x_j$ has been selected from the alias group.

(ii) No product term has been selected from the alias group.

Expressions for variances and covariances are obtained for both of the above cases. Expressions for certain relative efficiencies of alternative designs are also worked out.

HARTLEY, H. O. (Iowa State University, Ames)

Smallest composite designs for quadratic response surfaces—*In English*

Biometrics (1959) **15**, 611-624 (4 refs., 3 tables)

9.3 (4.3) *continued*

The last part of the paper deals with methods of obtaining the variance of the estimated response surface, for both to the above cases.

The author points out that designs usually recommended as fractional replicates *per se* are not necessarily the best when used in composite designs. Alternatives permitting the estimation of the largest possible number of coefficients in the second-order surface are developed and their efficiencies discussed.

(J. E. White)

MORLEY JONES, R. (Dept. of Genetics, Univ. of Birmingham) **9.1** (0.6)
On a property of incomplete blocks—*In English*
J. R. Statist. Soc. B (1959) **21**, 172-179 (6 refs.)

The author shows that balanced incomplete block designs have the property that any two orthogonal treatment contrasts have orthogonal inter- and intra-block components.

Matrices **D** and **Δ** are defined, the elements being 1 or 0, according as the *j*th observation is in respectively the *i*th block or *i*th treatment or not. The incidence matrix specifying the design of the experiment is given as the product of **Δ** and the transpose of **D**. The elements of this matrix specify the number of times that the *i*th treatment appears in the *j*th block. Diagonal matrices are formed from **D** and **Δ** times their transposes. The diagonals of these matrices give respectively the block sizes and the number of replicates of the treatments.

A block contrast sometimes called an inter-block contrast, and a treatment contrast are defined. A contrast which is orthgonal to all block contrasts is called an intra-block contrast.

In section 3 a condition for resolving a treatment contrast into inter- and intra- block components is given. It is then shown that the necessary and sufficient condition that any two orthogonal treatment contrasts have both inter- and intra-block components orthogonal is that a certain suffixed summation over blocks involving elements of the design matrix, block sizes and number of replications per treatment should be independent of any two treatments chosen. The necessary and sufficient condition for a treatment contrast to be resolvable into inter- and intra-block components, each estimating the same treatment contrast is that it should be a latent vector of a certain matrix **M**. The elements of this matrix are suffixed summations depending on elements of the incidence matrix, block sizes, and numbers of replicates of each treatment.

The latent roots of **M** are investigated and an expression is obtained for the efficiency factor of a design. A proof is given of Yates' theorem, stating that the sum of the relative losses of information on a set of orthogonal contrasts in a balanced arrangement is equal to the number of degrees of freedom confounded in each replicate.

Two examples of the use of the matrix **M** in investigating the properties of a design are given, a 3 by 2 factorial experiment, and a balanced incomplete block.

(T. I. McGlaughlin)

NORDSKOG, A. W. (Iowa State University, Ames) **9.4** (7.2)
Note on optimum group size for progeny tests—*In English*
Biometrics (1959) **15**, 513-517 (2 refs.)

Robertson [*Biometrics* (1957) **13**, 442-450] developed a solution for optimum number of sires to test in a progeny testing plan. His solution assumed a fixed total number of progeny to be tested in each generation, and one tested offspring from each dam. Rendel [*Biometrics* (1959) **15**, 376-381] extended the solution by Robertson to half-sib selection. The criterion for optimum number was, in each instance, the expected genetic superiority of the selected sires or progeny groups.

The author has considered the case in which each sire has several mates and each mate has several offspring. Equal numbers of mates per sire and equal numbers of offspring per dam (as well as a fixed total number of offspring) are assumed. The expected genetic superiority of the selected sires is expressed as a function of (*a*) the fraction selected, (*b*) the regression of breeding value of the tested sires on their progeny means, and (*c*) the standard deviation of the sire progeny means. The resulting formula for genetic superiority is recast into terms of (*a*) the fraction selected, (*b*) numbers of mates per sire and offspring per dam, and (*c*) variance components among full sibs, dams, and sires.

Two cases are examined, namely the optimum number of sires to test for fixed values of (*a*) the number of offspring per dam and (*b*) the number of mates per sire. Using values of 1000 total offspring, 5 sires selected, and heritability equal to 0·1, expected genetic superiority (and hence expected genetic gain) is maximum with only one offspring per dam. However, the loss is only 5 per cent. when offspring per dam is increased to 10. Similarly, the optimum number of mates per sire is 25, but the loss is only about 3 per cent. with 5 mates per sire. The author suggests that economic values and costs of the testing programme should be considered, in a practical solution.

(N. R. Thompson)

QUENOUILLE, M. H. (London School of Economics) **9.2** (9.3)
Experiments with mixtures—*In English*
J. R. Statist. Soc. B (1959) **21**, 201-202 (2 refs.)

Experiments with mixtures are factorial experiments where the total level of all factors may not exceed a certain level (and is usually restricted to this level).

Scheffé considered the interaction between two factors A and B as a quadratic regression coefficient and Quenouille, considering equal proportions of A and B together with a non-interacting factor C, notices that Scheffé's model leads to an AB interaction which is not proportional to the total level of A plus B; he says that this is " not completely additive ". He notices a similar effect when all interactions are present in a $2\times2\times2$ factorial experiment with the mixture restraint and the factors present in equal proportions. The author remarks that problems of design, analysis and interpretation differ as between Scheffé's model and a model where there is complete additivity.

(D. E. Barton)

RENDEL, J. M. (Sydney University, Australia) **9.4** (7.2)
Optimum group size in half-sib family selection—*In English*
Biometrics (1959) **15**, 376-381 (2 refs., 2 tables, 1 fig.)

The objective of this paper is to arrive at selection procedures which maximise the genetic superiority of selected half-sib groups. The author states that Robertson's [*Biometrics* (1957) **13**, 442-450] expression having to do with selecting by progeny testing may be applied to selecting half-sib groups for practical purposes, but that this is not strictly true. A formula for the correlation between the breeding value of a half-sibship and its mean score is presented, followed by a formula for genetic advance when selecting a half-sibship. Similar formulae are presented for the progeny test situation. Tabular values of the genetic superiority of selected half-sibships and of selected sires under varying conditions with N animals tested in all, n for each sire, and h^2 the fraction of variance that is genetic, are shown. Comparisons among the genetic superiorities for any fixed number tested and heritability value enable the reader to choose the optimum group size that should be used. The principal field of application of the results of this study is poultry breeding.

(J. A. Gaines)

SCHUMANN, D. E. W. & BRADLEY, R. A. (Univ. of Stellenbosch, S.A. & Viginia Polytechnic Inst.) **9.9** (7.2)
The comparison of the sensitivities of similar experiments: Model II of the analysis of variance—*In English*
Biometrics (1959) **15**, 405-416 (4 refs., 5 tables)

This paper is concerned with the analysis of data which arise from necessarily subjective appraisals of an attribute of sample units which may be ill-defined, such as quality, performance, preference, etc. The problem becomes one of choosing the scale of " measurement " (or experimental technique) which is most likely to disclose treatment differences when they exist, and the authors propose a test whereby the sensitivities of analyses on two sets of observations for the same set of treatments may be compared, the two sets of observations being referred to two different scales of appraisal, and the two parallel experiments being distinct and independent in probability. The development is limited to such pairs of experiments which are similar, that is, which allow the same degrees of freedom for their respective treatment mean-squares and the same degrees of freedom for their error mean-squares.

A brief review of the authors' two earlier papers is presented and in the present paper the test is extended to the comparison of sensitivities of experiments when Model II applies, and the previous tables are extended to give 95·0, 97·5, 99·0 and 99·5 percentage points of the ratio of two central variance ratios with the same degrees of freedom, 2 to 21, 30, 40, 60, 120 for the numerators and 2 (1 to 10, 15, 20, ∞) for the denominators.

For Model II analysis of variance let σ_{0j}^2 be the variance of treatment effects and let σ_j^2 be the residual or error variance for the jth experiment, $j = 1, 2$. Then the null hypothesis for each experiment is that $\sigma_{0j}^2 = 0$, and when real treatment differences exist, the variance-ratios of treatment mean square to error mean square are distributed as $F_j(1+k_j\sigma_{0j}^2/\sigma_j^2)$ with $2a_j$ and $2b_j$ degrees of freedom (a_j and b_j are integers or half-integers) where k_j is the number of observations for each treatment. The measure of sensitivity proposed is the ratio σ_{0j}^2/σ_j^2, and letting $\gamma_j = k_j\sigma_{0j}^2/\sigma_j^2$, the null hypothesis for the test of significance on sensitivities is that $\gamma_1/k_1 = \gamma_2/k_2$, and the test statistic is the ratio of the two variance-ratios. For similar experiments in general, approximate percentage points of the test statistic may be found from the given tables, and in the case of identical experiments exact percentage points are those in the tables. Also outlined is the procedure for finding confidence limits for the ratio $(1+\gamma_1)/(1+\gamma_2)$.

For the case of identical experiments a numerical example is presented of the procedures for making the test of significance and for finding confidence limits. The observations are on the quality of sheets of dried veneer, with scales varying respectively from 0 to 50 and 0 to 8 with 0 in each case indicating a sheet free of degrade. The analyses are two-way classifications (sheets and observers) with interaction, where the component of interest is the variance for sheets. The result of the comparison of sensitivities of the two experiments is non-significant, and 95 per cent. confidence limits are computed for the above-mentioned ratio.

(C. C. Thigpen)

483

SHRIKHANDE, S. S. (University of North Carolina, Chapel Hill) **9.1** (–.–)
The uniqueness of the L_2 association scheme—*In English*
Ann. Math. Statist. (1959) **30**, 781-798 (10 refs.)

The author considers an incomplete block design for v treatments in b blocks, each containing k (less than v) different treatments, such that each treatment occurs the same number of times in the design and such that any two different treatments occur together in either λ_1 or λ_2 blocks where λ_1 and λ_2 are different non-negative integers. It is shown that the " association scheme " of the design is of a particular type known as L_2 association scheme, if

(i) $v = s^2$, $s = 2, 3$, or $s > 4$
(ii) each treatment occurs with exactly $(2s-2)$ treatments in λ_1 blocks and with the remaining in λ_2 blocks
(iii) with respect to any two treatments which occur together in λ_1 blocks, there are exactly $(s-2)$ other treatments, each one of which occurs with these two treatments in λ_1 blocks
(iv) with respect to any two treatments which occur together in λ_2 blocks, there are exactly two other treatments which occur with each of these two treatments in λ_1 blocks.

The L_2 association scheme is explained by Bose and Shimamoto [*J. Amer. Statist. Ass.* (1952) **47**, 151-184].

It has also been shown that for the case $s = 4$, besides the L_2 association scheme defined above there is one more type of association scheme possible. An actual example corresponding to this other association scheme is also given.

Necessary conditions for the existence of a design with the L_2 association scheme and $v = b$ are given.

(S. Shrikhande)

WEARDEN, S. (Kansas State University) **9.4** (7.2)

The use of the power function to determine an adequate number of progeny per sire in a genetic experiment involving half-sibs—*In English*

Biometrics (1959) **15**, 417-423 (6 refs., 2 tables)

The author is particularly concerned with genetic experiments which involve large animals. Since the use of large animals in such experiments is expensive and time-consuming it is of utmost importance that the experiment accomplishes its desired aim. To do this the experimenter must know when he has an adequate number of observations per genetic group before he undertakes a genetic study concerning an estimate of heritability large enough to be of practical importance.

A common genetic group used in studies of large animals consists of paternal half-sibs. Such is the case in most progeny tests, and one of the most common methods of estimating heritability is a function of the paternal half-sib correlation. The author takes advantage of the fact that the test for this correlation (and hence heritability) can be put into the form of an analysis of variance for a completely randomised design under Model II. Using the easily determined power of this test, the author describes a procedure to determine the number of observations needed per group (of half-sibs) in order, with given power, to detect differences within groups. The formula stated requires an iterative solution that is simple to calculate and rapidly convergent. Two tables give the calculated numbers of progeny per sire for varying numbers of sires and different size estimates of heritability according to the probabilities of the Type I and Type II errors.

An example is presented that illustrates how an experimenter might actually apply the procedure in an experiment where he wished to estimate the additive genetic variance or conduct a progeny test with respect to a certain trait.

(R. J. Taylor)

AKAIKE, H. (Institute of Statistical Mathematics, Tokyo) **10.2** (8.9)

On the statistical control of the gap process—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **10**, 233-259 (11 refs., 9 figs.)

The gap process is defined as a strictly stationary discrete-time-parameter process $\{X_n(\omega);\ -\infty<n<+\infty\}$ where $X_n(\omega)=1$ and 0 with probability one and the gaps between successive 1's of the process are mutually independent and follow one and the same gap distribution $\{p_\nu;\ \nu=1, 2, \ldots\}$ with $L=\Sigma\,\nu P_\nu<\infty$.

This definition was introduced by the author in his previous paper [*Ann. Inst. Statist. Math., Tokyo* (1956) **8**, 87-94] as a model which represents the sequence of time points of renewals on the stationary flow. The variance-time curve of the process is defined as $\{(V(h),\ h);\ h=1, 2, \ldots\}$ where $V(h)$ represents the variance of $\sum_{i=1}^{h} X_{t+1}$ (the number of 1's in a time interval of length h).

Then $V(h)$ is represented as $V(h)=P\{2R(h)-h\}-h^2P^2$ where $P=1/L$, $R(h)=R(h-1)+\sum_{\nu=0}^{h-1} P_\nu$ with $R(1)\equiv 1$ and $P_\nu=\sum_{\mu=1}^{\nu} p_\mu P_{\nu-\mu}$ with $P_0\equiv 1$. In the present paper the asymptotic distribution of an estimate $\hat{V}(h)$ of $V(h)$ is given where $\hat{V}(h)$ is defined by replacing the p_ν's in the definition of $V(h)$ by $\hat{p}_\nu$'s, the observed relative frequencies of the gaps of length ν. By using the idea of the differentiable statistical function introduced by von Mises [*Ann. Math. Statist.* (1941) **12**, 263-267] the author obtains, with the aid of the generating function method, the result that under the condition $\sum_{\nu=1}^{\infty} \nu^2 p_\nu<+\infty$ $\sqrt{N}(\hat{V}(h)-V(h))$ tends asymptotically to the normal distribution with zero-mean and variance D_h^2 : N being the number of the independent observations of the gaps used to define the $\hat{p}_\nu$'s. This asymptotic result is checked by Monte Carlo methods for various types of $\{p_\nu\}$ and finite N's. The approximation is fairly good, even in some cases for $N=250$.

The results of the Monte Carlo verification and the formal analysis of the opening sections of the paper are used to construct a statistical control system for the gap process. The model is taken from a silk-reeling process and some numerical examples are given at the end of the paper.

The statistical control system is seen to centre upon the appropriate choice of the length of the intervals of observation (h) and the mean value of $\nu_t(h)$, certain rules are given and a summary procedure for setting up a statistical control system for a gap process. A warning is given in connection with a form of systematic bias that can be introduced by using the gap process with discrete time parameter as examined in this paper, to approximate a continuous time parameter process.

(H. Motoo)

BALAKRISHNAN, A. V. (University of California, Los Angeles) **10.7** (10.2)

On a characterisation of covariances—*In English*

Ann. Math. Statist. (1959) **30**, 670-675 (4 refs.)

Let $x(t)$ be a possibly-complex-valued stochastic process (random function). $x(t)$ is said to be of second order if the expected value of $[x(t)]^2$ exists. In this case the covariance function $F(s, t)$ of $x(t)$ also exists and is a non-negative definite Hermitian function. Let $m(s)$ be a complex valued function, defined for $-\infty<s<+\infty$. It is then only a trivial matter to show that $F(s, t)+m(s)\overline{m(t)}$, where the horizontal bar denotes the complex conjugate, is also a covariance function. This, however, is not true if one considers instead of $F(s, t)+m(s)\overline{m(t)}$ the function $F(s, t)-m(s)\overline{m(t)}$.

The author assumes that the process is stationary of the second order; this means that $F(s, t)=R(s-t)$ depends only on the difference $s-t$. He gives necessary and sufficient conditions which assure that $R(s, t)=R(s-t)-m(s)\overline{m(t)}$ is a covariance function. One of these conditions states that $m(t)$ shall be the Fourier transform of a function μ of bounded variation. The second condition involves the spectral distribution G corresponding to $R(t)$ and requires that the $L_2(dG)$-norm of the Radon-Nikodym derivative $d\mu/dG$ be less or equal to unity.

The author applies his result to the problem of estimating the mean value of a stochastic process.

(E. Lukacs)

CANSADO, E. (Inter-American Training Centre, Santiago) **10.0** (2.9)

The characteristic function as the mean of a stochastic process—*In Spanish*
Estadística Española (1958), **1**, 19-31 (5 refs., 5 figs.)

Stochastic processes are first defined as a special case of random functions. Starting with the simple concept of the random variable, the case is considered of circular or angular distributions, and the parameters of the uniform circular distribution are given.

In order to obtain circular distributions, a linear distribution is " wrapped " around a circle in a way similar to methods previously used by Lévy, Marcinkiewicz and Wintner for the case of a normal or Cauchy's distribution (normal " wrapped-up " distribution, angular case of Cauchy's distribution). The procedure for the " wrapping-up " is then graphically represented: formulae for the distribution function and the density function of the circular variate are also given.

Another interpretation of the " wrapped-up " distribution is made, imagining a circle which rolls without sliding on a straight line taking in each point of contact the mass of the corresponding point in the line. (Both the origins of the circle and its tangent are supposed to coincide.) The transformation of the linear uni-dimensional distribution into the circular bi-dimensional distribution is given in terms of the random variable.

The characteristic function of a real uni-dimensional random variable is defined in the usual way, later being also considered as the mean of a stochastic process, according to the definition. In order to analyse the process, its realisations or trajectories are examined as well as the probability distributions corresponding to different values of t. It is seen that to each particular value of the random variable there is a corresponding trajectory in the three-dimensional space which is the Cartesian product of the complex plane $(y+iz)$ and the straight line (t). This trajectory is an ordinary helix which forms a constant angle with the generators of its containing cylinder. All the trajectories are symmetric with respect to the axis y (real axis of the complex plane) and therefore have Hermitian symmetry. On the other hand, for each particular value of t, the random variable e^{it} has a bi-dimensional distribution in the complex plane, and this distribution is a " wrapped " circular one. For $t = 1$ the circular distribution is obtained by " wrapping " the linear distribution round itself. For $t = 0$ and any distribution of ξ, t has a singular distribution, with all the mass (or probability) concentrated at the origin. The corresponding circular " wrapped-up " distribution has all its mass concentrated in a point.

The linear distribution corresponding to $-t\xi$ is obtained by symmetry from the distribution of $t\xi$. Therefore the " wrapped " distribution obtained from the linear distribution of $-t\xi$ is the circular distribution obtained by symmetry with respect to the real axis of the complex plane from the circular distribution corresponding to $t\xi$.

(F. Azorín)

489

FIRESCU, D. (Bucharest University) **10.2** (4.8)

Estimation functions for the inverse transition probabilities of a simple stationary Markoff chain with a finite set of states—*In Rumanian*
Analele Univ. " Parhon " Seria St. Naturii (1959) **21**, 15-23 (5 refs.)

This is the third paper in a sequence by the author dealing with estimation properties of a particular kind of Markoff chain: it is assumed that the stochastic matrix of this type of Markoff chain is not decomposable.

The first paper [*Analele Univ. " Parhon "* (1959) **18**, 9-18] showed that the estimation functions corresponding to a subset of unknown transition probabilities belonging to a fixed row are asymptotically Gaussian. These functions become asymptotically efficient only in a case which was studied by Mihoc [*Bull. Math. Soc. Sci. Math.-Phys. R.P.R.* (1951) **1**, 449-456]. This case studied by Mihoc was where the particular subset under consideration coincides with the set of all transition probabilities of the fixed row.

In the second paper by the author [*Analele Univ. " Parhon "* (1958) **20**, 37-47] the results were extended to the case of an arbitrary subset of unknown transition probabilities. In this case, asymptotic efficiency is reached when the subset coincides with the set of all transition probabilities forming the stochastic matrix of the given Markoff chain.

In the third and present paper, analogous results are shown to hold true for an arbitrary subset of unknown inverse transition probabilities. The condition for this is that the characteristic equation has a simple and unique root of unit absolute value.

(R. Theodorescu)

FUCHS, A. (Strasbourg University, France) **10.1** (1.3)
A problem of attained times—*In French*
Publ. Inst. Statist. Paris (1958) **7**, 161-166 (3 refs.)

This paper considers a time homogenous discrete Markoff chain with a finite number (r) of states. The probability that the system passes from state (i) to state (j) in n consecutive steps is represented by P_{ij}^n. It is shown that the variable $T_i(G)$ is independent of n, when (G) is any group of states. The random variable $T_i(G)$ is termed "the attained time" of the group of states (G) on leaving state (i). The concept is particularly interesting when the group of states (G) is a final, or end group: the case where the group is reduced to a single element (j) is treated in detail.

The probability law of the random variable T_{ij} is given; T_{ij} being the attained time to reach state (j) from state (i). The probability that having left state (i) the system arrives at state (j) for the first time after n consecutive steps is denoted by Q_{ij}^n. After stating a theorem relating to T_{ij}, the author considers the expectation of T_{ij}, including the case where (j) is an absorbing state.

The paper concludes with an urn example consisting of random drawings with replacement from a set of balls of r different colours.

(W. R. Buckland)

491

GABRIEL, K. R. (Hebrew Univ. Jerusalem, and N. Carolina) **10.1** (1.3)
The distribution of the number of successes in a sequence of dependent trials—*In English*
Biometrika (1959) **46**, 454-460 (5 refs., 3 tables)

This paper considers a simple Markoff chain of $n+1$ events (either successes or failures) so that the probabilities p_1 and p_0 (being the probabilities that a success followed by respectively a success and a failure) are the only other parameters. The distribution of the number S of successes is studied with particular reference to the adequacy (or otherwise) of the normal limit for approximation when n is small [see Feller, *Introduction to Probability Theory and its Applications* **1** (1947), New York, Wiley].

The probability distribution function of S is obtained by summing over the joint probability distribution function of S and the number of runs (of success or failure) as given by David (1947). Lengthy expressions for the first four factorial moments of S are obtained (in terms of n, $d = p_1 - p_0$ $P = p_0/(1-d)$ and R, the possibility of the initial event being a success) by means of the method of characteristic random variables. The central moments under various simplifications are then given (for $R = P$ and neglecting terms in d^n). The latter case gives Markoff's values for the first two moments and simple expressions for the cumulant ratios.

Tables of exact and approximate values are given for $P = 0{\cdot}6$ and $0{\cdot}8$, $n = 7$ and 30 and $d = 0(0{\cdot}2)\ 0{\cdot}8$. When d is equal to zero the distribution is binomial: as d increases so do the variance and measures of skewness and kurtosis. The closeness of the approximations appear to the author to depend largely upon d being small and n large. An example shows that at least in one case the normal approximation using the large sample variance estimate is very close to the discrete exact distribution. A final comment made in the paper is that the rather tedious computation of the exact probability distribution is facilitated by tables of the logarithms of binomial coefficients [see Hald, *Statistical Tables and Formulas*: Table 14 (1952) New York, Wiley].

(D. E. Barton)

492

GHOSAL, A. (C.S.I.R., Jealgora, Bihar, India) **10.4** (1.1)

On the continuous analogue of Holdaway's problem for the finite dam—*In English*

Aust. J. Appl. Sci. (1959) **10**, 365-370 (4 refs.)

This paper applies Prabhu's method [*Quart. J. Math.* (1958) **9**, 183-188] of solving the integral equation for the stationary distribution function of a dam content (which arises in Moran's theory of dams) to a more complicated model suggested by Holdaway in which inputs have a negative exponential distribution [Moran, *Aust. J. Appl. Sci.* (1955) **6**, 117-130).

In Holdaway's scheme, the content $Z_t = 0, 1, 2, \ldots k$, of the dam at discrete times $t = 0, 1, 2, \ldots$ is given by the equation

$$Z_{t+1} = \begin{cases} \text{Min}\{Z_t+X_t,\ k+m\}-m & \text{for } Z_t+X_t \geqq l+m \\ \text{Min}\{Z_t+X_t,\ l+d\}-\text{Min}\{Z_t+X_t,\ d\} & \\ \qquad \text{for } Z_t+X_t < l+m & \end{cases}$$

where $l<k$, $d<m<k$; the $\{X_t\}$ are independent discrete inputs which flow into the dam in the intervals $(t,\ t+1)$ and have identical probability distributions $\{p_i\}$ $(i = 0, 1, 2, \ldots)$.

The processes $\{Z_t\}$ and $\{Z_t+X_t\}$ are Markoff chains, and equations for the stationary probabilities $R_i = \Pr\{Z_t+X_t = i\}$ $(i = 0, 1, 2, \ldots)$ in this model are given.

When X_t and Z_t take on continuous values, the distribution function of the inputs in the intervals $(t,\ t+1)$ being $\Pr\{X_t \leqq x\} = G(x) = \int_0^x g(t)dt$, with $t = 0, 1, 2, \ldots$ remaining discrete, the analogue of the previous process gives rise to the following integral equations for the distribution function: $H(y) = \Pr\{Z_t+X_t \leqq y\}$ of Z_t+X_t:

$$H(y) = \begin{cases} H_1(y) & \text{for } y<l, \\ H_1(y)+H_2(y) & \text{for } l \leqq y<k, \\ H_1(y)+H_2(y)+G(y-k) & \text{for } y \geqq k \end{cases}$$

where $H_1(y) = \int_d^{\text{Min}\,(d+y,\ d+l)} H(t)g(y+d-t)dt$, and

$$H_2(y) = \int_{l+m}^{\text{Min}\,(m+y,\ k+m)} H(t)g(y+m-t)dt.$$

For inputs X_t having a negative exponential density function $g(x) = \mu/\exp(\mu x)$, the author makes use of the transformation $\phi(y) = 1/\exp(\mu y)\{1-H(k+m-y)\}$ to solve these integral equations. The stationary distributions of Z_t+X_t and of the dam content Z_t are obtained explicitly.

(J. Gani)

GILBERT, E. J. (Sandia Corporation, Albuquerque) **10.2** (10.1)

On the identifiability problem for functions of finite Markoff chains—*In English*

Ann. Math. Statist. (1959) **30**, 688-697 (3 refs.)

Suppose a stochastic process is known to be a stationary irreducible Markoff chain with a finite number of states. Associated with such a Markoff chain is a matrix of transition probabilities; that is to say, the probabilities that the process passes from state i to state j, for all $i \neq j$. For some reason the states of the process cannot be directly observed, but the states of the process are divided into groups, such that it is possible to identify the group from which an observation came. The observable process is again stationary; that is to say the transition probabilities do not change with time: but it is not in general a Markoff chain. Hence the transition probabilities depend on the past history of the process.

A given transition matrix, together with the function which partitions the states into groups, uniquely determines the distribution of the observable process. For any given partition function, there may be more than one transition matrix which gives rise to the observable process. Hence even if the entire distribution of the observable process is known, the matrix of transition probabilities for the original process cannot be determined. The general problem studied in this paper is to specify the characteristics of the observable process which make it possible to identify the class of Markoff chains (transition matrices) which could give rise to it. A characterisation of the number of identifiable parameters is obtained under a specified regularity condition. Some further conjectures are also noted.

(D. G. Chapman)

HAIGHT, F. A. (University of California, Los Angeles) **10.4** (–.–)
Queuing with reneging* —*In English*
Metrika (1959) **2**, 186-197 (10 refs.)

(* *deserting or defecting*)

This article deals with a generalisation of the usual queuing theory, attempting to take into consideration the rational behaviour of an individual queuing up.

The author starts with the assumption of a queue having a Poisson input, and an exponential service time distribution. An individual arriving at this queue can behave in the following ways: (*a*) baulk, i.e. not join, (*b*) join and subsequently renege, i.e. go away or desert, and (*c*) join and remain for service. Actually (*b*) represents the generalisation in contradistinction to the models used up to now. Baulking then is the special case of deserting on arrival.

Three aims can be distinguished:

(i) To give a decision scheme for reneging, by means of which an individual can behave rationally in uncertainty;
(ii) To investigate the expected result of such a scheme of decision for the individual, and
(iii) To describe the behaviour of a queue in which such schemes are operative among all queuers.

The author stresses the fact that in his paper these problems have not reached a final solution: he also suggests that it might be possible to arrive at a better formulation.

All the following models contain W, the maximum tolerable waiting time. Finding a decision scheme which reflects the rational behaviour in the queue, is reduced to the definition of a function $\phi(t)$ (t = time) which should be " as good as possible ". $\phi(t)$ is to be selected from a class of functions with the following properties: non-decreasing, $\phi(0)<0$, $\phi(\infty)=\infty$, and the queuer will desert when $n(t)$ and $\phi(t)$ intersect; $n(t)$ is a step function indicating the number of departures from the queue as a function of time. In this article, only a special class of functions ϕ is considered, namely a one-parameter family of hyperbolas. Choosing the hyperbolas out of the class of functions, ϕ, is not based on a criterion of optimalisation, but on additional assumptions concerning the individual's behaviour.

Having selected a certain function $\phi(t)$ and, furthermore, having determined the value of the time used for waiting and service as well as the value the individual attributes to being served in his utility scale, we can find an utility function U—thus entering the sphere of the decision problem itself. The practical usefulness, however, is limited by the fact that the connection between U and any particular $\phi(t)$ is not easy to obtain.

In the last section the state probabilities for a queue with baulking and deserting are computed. The main difference characterising this approach compared to the usual type, is the use of an additional function, $\Pi(x)$, having the property that the probability of one deserter in Δt is $\Pi(x).\Delta t$ when the queue contains x members. Thus the author succeeds in finding explicit solutions for the asymptotic state probabilities.

495 (F. Ferschl)

HATORI, H. (Department of Mathematics, Tokyo Inst. of Tech.) **10.5** (–.–)
A note on the entropy of a continuous distribution—*In English*
Kodai (*Tokyo Inst. Tech.*) (1958) **10**, 172-176 (4 refs.)

Following the definition of the entropy of a continuous distribution (after Shannon), Goldman has given an explanation on the basis of the discrete case and from some postulates Reich has derived an expression for the information rate. In this paper the author gives another set of postulates (which are plausible) from the discussions for the discrete case by Shannon and Khinchine. It is shown that the entropy $H(f)$ of an n-dimensional continuous distribution with probability density function $f(x_1, \ldots, x_n)$ which satisfies the set of postulates must be expressed by

$$H(f) = -\lambda \int \ldots \int f(x_1, \ldots, x_n) \log f(x_1, \ldots, x_n) dx_1 \ldots dx_n,$$

where x is a positive constant.

The postulates are:

(i) The entropy $H(f)$ of a continuous distribution with density function f is determined by f alone.
(ii) The entropy $H(f)$ of the distribution with density f is smaller than the entropy $H(g_s)$ of the distribution with density g_s which is uniform on a subset S of the n dimensional Euclidian space, if S contains the carrier of f and $f \neq g_s$ (almost everywhere).
(iii) Let $\phi(x_1, \ldots x_k)$ be the probability density function of the first k random variables among the n variables and $\psi x_1, \ldots x_k(x_{k+1}, \ldots, x_n)$ the conditional probability density function of the remaining $n-k$ variables under the condition that the first k variables have respectively taken the values $x_1, \ldots, x_k$. Then

$$H(f) = H(\phi) + \int \ldots \int H(\psi)\phi(x_1, \ldots, x_k) dx_1, \ldots dx_k.$$

(iv) If the number of different values of f is finite, $H(f)$ depends only on these values of f and the Lebesgue measures of the sets of those points at which f is a constant. $H(f)$ does not depend on the dimension number n.

(I. Higuti)

JOSHI, D. D. (Panjab University, Chandigarh, India) **10.5** (4.1)
Information in mathematical statistics and in communication theory—*In French*
Publ. Inst. Statist. Paris (1959) **8,** 82-159 (39 refs.)

This review paper consists of an introduction and four sections (called chapters) as follows:

(i) Fundamental theory of information theory: discrete case
(ii) The problem of binary coding.
(iii) The fundamental theorem: general case.
(iv) Information in mathematical statistics.

Since the original work of Shannon the theory of information has made rapid progress and now has many fields of application. In the author's opinion two papers by Khintchine [*Uspekhi Mat. Nauk.* (1953) **8**, 3 and (1956) **11**, 17] provide the best exposition of Shannon's theory.

In the first chapter, the present author introduces the two fundamental concepts of entropy and capacity and shows that they are bound up with different problems of coding. This leads to a consideration of binary coding which is studied in the second chapter: the original work of Hanning [*Bell Syst. Tech. J.* (1950) **29**, 147] is cited together with two theorems due to Zaremba [*J. Lond. Math. Soc.* (1952) **27**, 242]. The chapter concludes with two new inequalities on the problem of optimal coding.

The third chapter envisages the possibility of generalising Shannon's theory to abstract spaces. After considering the concept of a transmission line in measurable space, and having defined its capacity, a theorem is given which is analogous to Shannon's fundamental theorem.

The final chapter is devoted to an examination of the concept of information in mathematical statistics. Many authors have tried to show the unity between the concept of information due to Fisher and that of Shannon-Wiener. In particular the author refers to Schutzenberger's paper [*Publ. Inst. Statist. Paris* (1954) **3**, 3] and Barnard's work on an axiomatic theory [*J. R. Statist. Soc.* B (1951) **13**, 46]. The author re-examines the problem from a somewhat different point of view. Finally, two problems are studied: a test of hypothesis and discrimination. It is shown that the Shannon-Wiener theory can be used for the former and the " distance " function given by Chernoff [*Ann. Math. Statist.* (1952) **23**, 493] for the latter problem.

(Mlle. Gervaise)

KAC, M. & SLEPIAN, D. (Cornell Univ., Ithaca, N.Y. and Bell Telephone Labs., Murray Hill, N.J.) **10.1** (2.9)
Large excursions of Gaussian processes—*In English*
Ann. Math. Statist. (1959) **30**, 1215-1228 (7 refs.)

The probability distribution is sought of the interval a Gaussian stochastic process $x(t)$ attains above a level $x = a$ on a single excursion; this interval is called the " recurrence time " of the level. The process has zero mean, unit variance, and an autocovariance $\rho(\tau)$. Results are given for the limits $a \to \infty$ and $a \to -\infty$, with main attention paid to the former. Previous derivations required $x(t)$ to be twice differentiable; here differentiability once is all that is necessary.

The distribution sought is $\Pr\{x(t') > a, 0 \leqq t' \leqq t \mid x(0) = a, x'(0) > 0\}$. The specification of conditions on both $x(t)$ and its derivative $x'(t)$ at $t = 0$ involves certain difficulties in the definition of conditional probabilities, which are discussed at length. To define a conditional probability density like $p(\xi \mid x(0) = a)$, $\xi = x'(0)$, one must form the density $p(\xi \mid A)$, where A is some condition of finite probability $p(A)$, and one must then pass to a limit in which the condition A reduces to $x(0) = a$. The form of the conditional density depends on this limiting process. In the " vertical window " definition, A is " $a \leqq x(0) \leqq a + \delta$ ", and one lets $\delta \to 0$, obtaining a normal distribution for ξ with zero mean and variance $\alpha = |\rho''(0)|$. In the " horizontal window " definition, A is " $x(t) = a$ for $0 \leqq t \leqq \delta$ ", and when $\delta \to 0$ one finds that $p(\xi \mid x(0) = a) = (|\xi|/2\alpha) \exp(-\xi^2/2\alpha)$. Other choices of A lead to still other forms. The empirical definition, in which one measures $\xi = x'(t)$ at all points where $x(t) = a$ in a realisation of the process and forms the probability density of the results, is equivalent to the horizontal window definition. The horizontal window definition is related to Smoluchowski's definition of the mean recurrence time of a stochastic process.

The distribution of recurrence times is found in terms not of $x(t)$ but of a scaled process $\Delta(t, \theta) = [x(\theta t) - a]/\theta$, where θ is the expected length of intervals during which $x(t) > a$. For $a \gg 1$, $\theta \sim a^{-1}(2\pi/\alpha)^{1/2}$. By the method of characteristic functions it is shown that when $a \to \infty$ the sample functions of $\Delta(t, \theta)$ become a family of random parabolas. By means of the inclusion and exclusion formula the probability density $Q(T)$ for the recurrence time T of $\Delta(t, \theta)$, given $\Delta(0, \theta) = 0$, $\Delta'(0, \theta) > 0$, is derived. When the two conditions at $t = 0$ are imposed in the horizontal window sense, $Q(T) = (\pi T/2) \exp(-\pi T^2/4)$, and when they are imposed in the vertical window sense, $Q(T) = \exp(-\pi T^2/4)$, $T > 0$; $Q(T) = 0$, $T < 0$. The probability density for $a \to -\infty$ is $Q(T) = 2/\exp(2T)$ under both horizontal window and vertical window conditions, where now $\theta \sim 2\pi\alpha^{-1/2} \exp(a^2/2)$. An appendix provides formulas for means and covariances of general multivariate normal conditional probability densities.

(C. W. Helstrom)

KALLIANPUR, G. (Michigan State University) **10.2** (6.1)

A problem in optimum filtering with finite data—*In English*
Ann. Math. Statist. (1959) **30**, 659-669 (3 refs.)

The first problem treated is to find a filter to estimate the value at time $T+T_1$ of a signal $x(t) = Vg(t)$ on the basis of observations of the sum $z(t)$ of signal and noise over the finite interval $0 \leqq t \leqq T$. The estimate sought is the one that minimises the expected square deviation. $g(t)$ is a known continuous function that does not vanish everywhere in $0 \leqq t \leqq T$, and its amplitude V is a random variable with zero mean and a known probability density $h(V)$. The "noise" is Gaussian with a covariance function $R(t, s)$ that is continuous in the square $\{0 \leqq t, s \leqq T\}$. It is assumed that the integral equation $\int_0^T R(t, s)p(s)ds = g(t)$ has a square integrable solution. The best estimate of $x(T+T_1)$ is its conditional expected value, given $z(t)$ over $0 \leqq t \leqq T$; this estimate is derived by using an expansion of $z(t)$ in eigen-functions of the covariance $R(t, s)$. It turns out to be just $g(T+T_1)$ times the conditional expectation of the amplitude V, and this depends on the observations $z(t)$ only through the statistic $\int_0^T z(t)p(t)dt$.

The method is then extended to estimate $x(T+T_1)$ when $x(t)$ is the sum of a finite number of signals $V_i g_i(t)$ with known shapes $g_i(t)$ and amplitudes V_i described by a known joint probability density function, under the proviso that the $g_i(t)$ are linearly independent. Again one substitutes into $x(T+T_1)$ the best estimates of the amplitudes V_i. The case where $x(t)$ is an arbitrary second-order continuous stochastic process with known covariance $M(t, s)$ is treated as a limiting case of this problem. The functions $g_i(t)$ are replaced by the eigen-functions of the covariance $M(t, s)$, and one estimates the coefficients of the expansion of $x(t)$ in a series of those eigen-functions. The result in this case is necessarily left in the form of an infinite sum. The expected square-error is calculated for the case where $x(t)$ is the sum of a finite number of signals. The author discusses the relation of his treatment to an earlier approach to the problem given by Laning and Battin, emphasising by an example the necessity of careful handling of the limit processes involved. The connection with the problem of estimating the coefficients of a linear regression is pointed out.

(C. W. Helstrom)

LOMNICKI, Z. A. & ZAREMBA, S. K. (Boulton Paul Aircraft Ltd., Wolverhampton) **10.6** (–.–)

Bandwidth and resolvability in statistical spectral analysis—*In English*
J. R. Statist. Soc. B (1959) **21**, 169-171 (7 refs.)

The purpose of this paper is to refute the latest objections of Grenander to the author's work, and to clarify the distinction between resolvability, bandwidth, and bias. In their original paper [*J. R. Statist. Soc.* B (1957) **19**, 13-37; for discussion, see pp. 47-63], the authors referred to Grenander's "uncertainty principle" [*Ark. Mat.* (1951) **1**, 503-531], namely the theorem that the product $\Delta_1\Delta_2$ has a positive lower bound, where Δ_1^2 is a second moment of the weighting function used to average the periodogram, and Δ_2^2 is the limit of the product of the sample size and the variance of the estimator. They criticised Grenander's interpretation of Δ_1 as a measure of "lack of resolvability", and of Δ_2 as being "asymptotically proportional" to the standard deviation of the estimator. The authors interpreted "resolvability" as being the ability to estimate differences in spectral densities at neighbouring frequencies; and since (according to their criterion) the derivative of the "best" estimator of the density is the "best" estimator of the derivative, they concluded that resolvability and statistical reliability are not antagonistic.

Grenander contributed briefly to the discussion on the above paper, and later expanded his remarks—which were written before the authors' reply to the discussion was published—[*J. R. Statist. Soc.* B (1958) **20**, 152-157]. He rejected the authors' heuristic argument that the uncertainty principle as stated would lead to the conclusion that the periodogram itself is a good estimator, since the conditions of the theorem are not satisfied by the Dirac weight function which is then required. He pointed out that the "principle" could be proved in several forms, and reaffirmed that "resolvability (bandwidth, bias) and statistical realiability are antagonistic". Finally, the authors' results applied to estimators which could only be constructed when the spectrum was known, which does not correspond to any practical problem.

In the present paper, the authors again distinguish the concepts of resolvability, bandwidth and bias. They had never disputed the fact that bandwidth and variance oppose one another and had introduced the Dirac function only as a limiting case. Grenander now seemed to be using "bandwidth" in a sense different from that of his 1951 paper; his final point was not relevant, and was the more surprising since he had himself [with Rosenblatt, *Statistical Analysis of Stationary Time Series* (1957), Wiley] proposed to optimise estimators by a method depending on the true spectral density.

The authors point out an error in Grenander and Rosenblatt's derivation of the bias of the truncated periodogram.

(C. L. Mallows)

KRAMER, H. P. (Bell Telephone Labs., Murray Hill, N.J.)
Symmetrisable Markoff matrices—*In English*
Ann. Math. Statist. (1959) **30**, 149-153 (2 refs.)

10.1 (10.0)

This paper deals with the equilibrium or stationary probabilities of Markoff processes in continuous time having finitely many state values. The state probabilities of the process, $\{p_i(t)\}$, satisfy the differential equations

$$dp_i/dt = \sum_{j=1}^{N} Q_{ij}p_j(t), \quad i = 1, \ldots, N.$$

The coefficients $\{Q_{ij}\}$ are independent of time, and are the transition rates between states; they satisfy the condition $\Sigma_{j=1}^{N} Q_{ij} = 0$.

The equilibrium probabilities of the process are $\pi_i = \lim_{t\to\infty} p_i(t)$; if they exist they satisfy $\Sigma_{j=1}^{N} Q_{ij}\pi_j = 0$, with $\Sigma_{j=1}^{N} \pi_j = 1$. Equilibrium probabilities find much use in applications; for examples see Feller, *An Introduction to Probability Theory*, Chapter 17. The author observes that, in general, the equilibrium probabilities cannot be explicitly represented as simple functions of the Q_{ij}'s.

Exceptions are the birth-and-death processes; in these processes transitions occur only from i to $i+1$ (with rate λ_i) or from i to $i-1$ (with rate μ_i), and

$$\pi_i = \frac{\lambda_1\lambda_2\ldots\lambda_i}{\mu_2\mu_3\ldots\mu_{i+1}}\,\pi_1 \quad i = 1, \ldots, N.$$

This paper discusses a class of Markoff matrices $\mathbf{Q} = [Q_{ij}]$ which includes the birth-and-death matrices and allows a representation similar to that immediately above for the birth-and-death probabilities. This is the class of $1_2(\mu)$-symmetric matrices with connected graph. The required definitions follow. $\mathbf{Q}$ is an N by N matrix. The graph $G(Q)$ associated with $\mathbf{Q}$ consists of vertices $V_1, \ldots V_N$ and of directed line segments joining certain of them. The directed line segment from V_j to V_i, $\overline{V_jV_i}$, belongs to $G(Q)$ if and only if $Q_{ij}\neq 0$. Denote by $[k_1k_2\ldots k_s]$ the product of matrix elements $Q_{k_1k_2}\cdot Q_{k_2k_3}\cdots Q_{k_{s-1}k_s}$. V_i is connected to V_j if there exist indices $k_1, k_2, \ldots k_s$ such that $[jk_1\ldots k_si]\neq 0$. A graph G is connected if (*a*) in case $\overline{V_iV_j} \in G$, then $\overline{V_jV_i} \in G$, and (*b*) every pair of vertices is connected by at least one chain of lines. Let μ be a sequence of N positive numbers; $1_2(\mu)$ is an inner product space of sequences of N complex numbers, with inner product

$$(x, y) = \sum_{i=1}^{N} x_i\bar{y}_i\mu_i.$$

Every N by N matrix is an operator on $1_2(\mu)$. A matrix $\mathbf{A}$ is $1_2(\mu)$ symmetric if $(\mathbf{A}x, y) = (x, \mathbf{A}y)$ for every x and y in $1_2(\mu)$. It is shown that a matrix with non-negative off-diagonal elements is $1_2(\mu)$ symmetric if and only if the

KRAMER, H. P. (Bell Telephone Labs., Murray Hill, N.Y.)
Symmetrisable Markoff matrices—*In English*
Ann. Math. Statist. (1959) **30,** 149-153 (2 refs.)

10.1 (10.0)
continued

"loop condition" holds: the product of matrix elements around a closed loop is independent of the direction of loop traversal. Furthermore, if the above $1_2(\mu)$ symmetric matrix has graph consisting of a single connected set, then there is a unique sequence μ (with respect to which the matrix is $1_2(\mu)$ symmetric) such that $\Sigma_{i=1}^{N} \mu_i^{-1} = 1$, $\mu_i > 0$. The equilibrium probabilities are given by $\pi_i = \mu_i^{-1}$, and are shown to be given by a quotient of products of matrix elements very similar to that for the birth and death processes.

Rate of approach to the equilibrium probabilities is discussed for $1_2(\mu)$ symmetric matrices. All characteristic values of such matrices are non-positive, zero being the largest value. It is shown that the difference between $p_i(t)$ and π_i is maximised by a function of t and the initial probabilities $\{p_i(0)\}$. An example is given.

(D. P. Gaver, Jr.)

MORAN, P. A. P. (Australian National University, Canberra) **10.9** (10.1)

The survival of a mutant gene under selection—*In English*

J. Aust. Math. Soc. (1959) **1**, 121-126 (7 refs.)

The author discusses the problem of survival of a single mutant in a finite genetic population of $2n$ haploid individuals in which generations overlap, the mutant gene having a selective advantage.

Selection is first assumed to act through the varying of lifetime distributions, the mutant gene a having a negative exponential distribution with parameter λ_1^{-1}, while the non-mutant gene A has a similar distribution with different parameter λ_2^{-1}]; individuals born as replacements are chosen at random from the gametic output of the previous generation so that the probability of their being a or A is $k(2n)^{-1}$ or $(2n-k)(2n)^{-1}$. The state of the population is then given by the number $k(= 0, 1, 2, \ldots, 2n)$ of genes a at those discrete times at which a birth-death event occurs, and the Markoff chain embedded in the continuous process is defined by the transition probabilities

$$p_{k,\,k-1} = \lambda_2 k(2n-k)(2n)^{-1}\{\lambda_2 k+\lambda_1(2n-k)\}^{-1},$$
$$p_{k,\,k+1} = \lambda_1 k(2n-k)(2n)^{-1}\{\lambda_2 k+\lambda_1(2n-k)\}^{-1},$$
$$p_{kk} = 1-p_{k,\,k-1}-p_{k,\,k+1}.$$

If, however, selection is introduced by the relative reproductive powers of a and A, these being in the ratio μ_1/μ_2 so that the probability of birth of a gene a or A is $\mu_1 k\{\mu_1 k+\mu_2(2n-k)\}^{-1}$ or $\mu_2(2n-k)\{\mu_1 k+\mu_2(2n-k)\}^{-1}$ while the lifetime distributions are the same negative exponential, the embedded Markoff chain of the process has transition probabilities also of the form shown above, with μ_1, μ_2 replacing λ_1, λ_2 respectively.

Since the ratio $p_{k,\,k+1}/p_{k,\,k-1}$ in both models is constant, being λ_1/λ_2 and μ_1/μ_2 respectively, the problems are seen to be identical. On embedding a secondary chain in the previous ones by considering only those jumps for which k changes its value, so that $p_{k,\,k-1} = 1-\pi$, $p_{kk} = 0$, $p_{k,\,k+1} = \pi$, where $\pi = \lambda_1/(\lambda_1+\lambda_2)$ or $\mu_1/(\mu_1+\mu_2)$, the models reduce to the classical gambler's ruin problem.

Writing $\pi = (1+\alpha)(2+\alpha)^{-1}$ where α is the selective advantage, the probability P_{2n} of survival of the gene a is thus seen to be

$$P_{2n} = (1+\alpha)^{2n-k_0}-(1+\alpha)^{2n}/1-(1+\alpha)^{2n}$$

k_0 being the initial number of genes a. For $\alpha = O(n^{-1})$ and n large, $P_{2n}\sim\alpha k_0(1-e^{-2\alpha n})^{-1}$, a result similar to Fisher's [*The Genetical Theory of Natural Selection* (1930) Oxford] approximation $P_{2n} = 2\alpha(1-e^{-4\alpha n})^{-1}$ for $k_0 = 1$ when generations do not overlap.

A comparison is made with previous results for associated models, and the need for a solution to the parallel diploid population model stressed.

(J. Gani)

NASR, S. K. (University of Alexandria) **10.4** (2.5)

On some problems of machine interference—*In English*

J. R. Statist. Soc. B (1959) **21**, 106-113 (10 refs.)

After a brief discussion of the work by Khinchine, Ashcroft, Palm, Naor, Benson, and Cox, the author considers Palm's solution as quoted by Feller [*Introduction to Probability Theory* **1** (1957), New York, Wiley]. For the case when m machines are attended by r operatives, an expansion is given for the servicing factor which is shown to be consistent. The expression for machine efficiency given by Naor in Poisson functions is then converted to binomial and Poisson cumulative functions: the author remarks that this may ease computation. An expression is given for the probability that there are some machines waiting to be serviced.

The author goes on to discuss the numerical results found in the literature and states that they may not seem to be compatible; an example is given. He then finds a limiting value of the machine efficiency as the number of machines increases indefinitely, while the number of machines attended by an operative remains fixed.

Finally, the effect on a loss function of assigning ancillary work to operatives is discussed. The loss function is used to obtain lower and upper bounds for the number of operatives required.

(T. I. McGloughlin)

ONICESCU, O. (Bucharest University) **10.0** (1.0)

On processes of a field of events and on the corresponding random function—*In French*
Bull. Math. Soc. Sci. Math. Phys. R.P.R. (1958) **2**, 95-108.

The aim of this paper is the construction of a stochastic process by the synthesis of its two basic components: a somatic process, and a measure. The somatic process is a non-stochastic process of a field of events depending upon time in a determined manner. To this set-theoretical aspect of the somatic basis of the process, a numerical aspect is associated considering certain numerical functions determined under suitable conditions. The measure, defined on the events of that field, is also varying in time, yielding the probabilities of the process.

For a finite field K, the events of the process are formed by associating every event A of K to a time instant t, starting from an initial moment. The elementary events of the process are constituted by temporally ordered sets of the elementary events, corresponding (according to a determined law) to every time instant t. The fundamental events of the process are formed by associating an event A of the field to a determined value of t, while the total event is associated to each other event. Events, formed by associating an event A to a time instant t, while an elementary event is associated to the initial moment, are also characterised. A set-theoretical Chapman composition law is outlined. By making use of this composition law, a system of probabilities is associated to the field yielding in this way a simple Markoff chain. The set functions defined on the non-probabilised event-field verify a non-linear functional equation recalling the equation of branching chains.

In the second part of the paper, starting from an event-field K defined on a basic space, a somatic process in time is constructed, by a method generalising the previous one. This becomes possible by making use of an integral operation concerning set-functions with values in a set-field. In this manner we are led to an event-composition law and similarly to the composition law of a system of probabilities.

(R. Theodorescu)

ONOYAMA, T. (Kyushu University, Japan) **10.6** (10.0)

Note on random distributions—*In English*
Mem. Fac. Sci. A, *Kyushu* (1959) **13**, 208-213 (5 refs.)

The random distribution introduced by Ito [" Stationary random distributions," *Mem. Coll. Sci., Kyoto*, A (1954) **28**] and Gelfand [" Generalised random processes," *Dokl. Acad. Nauk. SSSR* (1955) **103**] is an H (Hilbert) space-valued continuous linear functional defined on the space of basic functions D; where D is the set of all complex-valued C^{∞}-functions defined over $R(-\infty, \infty)$ and with compact carriers.

The stationary stochastic process finds its natural extension in the random distribution and the spectral representation of the stationary process has its analogue for the stationary random distribution. This spectral representation of the stationary random distribution is validated through the Fourier transforms of the basic functions.

Thus the extension of the space of basic functions will enable us to extend the definition of the stationary distribution. In the paper by Gelfand & Silov [" Fourier transforms of rapidly increasing functions and questions of uniqueness of the solution of Cauchy's problem," *Uspekhi Math. Nauk.* (1953) **8**] a foundation for the Fourier transforms of rapidly increasing functions was given. In the present paper the author defines the random distribution on the space $K_p(p>1)$, the space of functions $\phi(x)$ which satisfy the inequality $|\phi(x)| \leqq C_1 \exp(-C|x|^p)$ for some constants C_1 and C (depending on the function) together with each of their derivatives. The notion of the stationary distribution is then extended by using the results of Gelfand and Silov. The possibility is suggested of applying the results to the non-homogeneous stochastic linear translatable equations discussed by the same author [" Linear translatable stochastic functional equations and random distributions," *Mem. Fac. Sci.* A, *Kyushu* (1956) **10**].

(H. Akaike)

PRAPORGESCU, N. (Institute of Mathematics, Bucharest) **10.1** (1.0)
On forming chain probabilities—*In Rumanian*
Studii şi cercetări mat. (1958) **9**, 439-480 (7 refs.)

This paper deals with the asymptotic behaviour concerning stationary and non-stationary Markoff chains and chains with complete connections (notion introduced by Onicescu and Mihoc). This is done by making use of certain special systems of integral equations studied by the author in a previous paper [*Bul. ştiinţific al Acad. R.P.P., Secţiunea de ştiinţe matematice şi fizice* (1956) **8**, 549-616] as well as of the characteristic function method.

For the simple stationary chains, the general form of the transition probabilities is deduced. By making use of certain properties of the roots of the corresponding characteristic determinant the asymptotic values of the transition probabilities are deduced. In this way results given previously by Fréchet were rediscovered. Moreover, these results hold good for continuous and multiple chains. Furthermore, under suitable conditions the non-stationary chains have a similar asymptotic behaviour as in the stationary case. Analogous results are also deduced for chains with complete connections. The last part deals with slowly convergent distribution functions ; it is shown that the expression of the corresponding asymptotic characteristic function generalises to the Poisson distribution function.

As starting point, the author took the monograph of Onicescu and Mihoc " Les chaînes de variables aléatoires. Problèmes asymptotiques," *Études et recherches de l'Académie Roumaine* (1943) **14**.

(G. Sîmboan)

PRAPORGESCU, N. (Inst. Math., People's Republic Academy, Bucharest) **10.1** (–.–)
On temporally variable probabilities forming a chain—*In Russian*
Rev. Math. Pures Appl. (1959) **4**, 403-423 (7 refs.)

This paper is the sequel of a previous one published by the author in *Studii şi cercetări Matematice* (1958) **9**, 439-480. Non-stationary simple Markoff chains with discrete as well as with continuous set of states are investigated. By means of similar methods as used in the quoted paper, expressions for the solutions of the corresponding Chapman-Kolmogoroff equations are given. These expressions are used in order to characterise the asymptotic behaviour. It is stated by the author that the results hold true for multiple chains.

(R. Theodorescu)

RÉNYI, A. (Eötvös University, Budapest) **10.1** (2.5)

On serial and parallel coupling of autoclaves and on the theory of mixing—*In Hungarian*
Publ. Math. Inst. Hung. Acad. Sci. (1959) **4**, 155-165 (1 ref.)

The problem discussed in this paper arises from a situation where a material is mixed in an autoclave. The mixing is called ideal if the distribution of the time which a particle spends after some moment t in the autoclave does not depend on the length of the time which the particle has already spent there. This condition means that the distribution of the length of time which a particle spends in the autoclave is an exponential distribution. The author deals with the following problem: how to switch a given number of autoclaves if we want to maximise the percentage of the particles which remain more than t_0 hours in the system of autoclaves and if we want a given amount of material to flow through the autoclaves during an hour.

In the second part of the paper the problem of non-ideal mixing is discussed. The ratio $i = D^2(\xi)/M^2(\xi)$ is called by the author the "intensity of mixing": ξ is equal to the time which a particle spends in the autoclave, $M(\xi)$ is the mean value of $D^2(\xi)$ the variance of the random variable ξ. In the case of ideal mixing $i = 1$ and if there is no mixing at all, then $i = 0$. If the probability that a particle leaves the autoclave after spending a certain amount of time (x) there is a monotonically non-decreasing function of x, then the mixing is called effective. It is shown that in case of effective mixing $\log 1/\{1-P(\xi<x)\}$ is a convex function, which implies $i \leqq 1$.

(P. Révész)

509

RÉVÉSZ, P. (Eötvös University, Budapest) **10.0** (1.5)

A generalisation of the zero-one law—*In English*
Ann. Univ. Sci. Budapest. Sect. Math. (1959) **2**, 111-113 (4 refs.)

The zero-one law and the Borel-Cantelli lemma are well-known in the theory of probability. The validity of these theorems is usually proved only for mutually independent random variables or—in the case of the Borel-Cantelli lemma—for mutually independent events. In this paper the following interesting question is posed: is the condition of independence necessary or not in these theorems? Erdős & Rényi in their paper "On Cantor's series with convergent $\Sigma \frac{1}{q_k}$," [*Ann. Univ. Sci. Budapest.* Sect. Math. (1959) **2**, 93-109] proved that the assertion of the Borel-Cantelli lemma remains true if we suppose only that the events are pairwise independent. The author deals with the corresponding problem in the case of the zero-one law. It is shown by a simple counter-example that the assertion of the zero-one law is not true generally for pairwise independent random variables. However, the following theorem is valid: Let $\{\Omega, \mathcal{S}, \mathcal{P}\}$ be a probability field and let $\xi_1, \xi_2, \ldots$ be an ergodic Markoff-chain. Let $\mathcal{S}_n$ denote the smallest Borel-field with respect to which $\xi_n, \xi_{n+1}, \ldots$ are measurable and put $\pi\mathcal{S}_n = \mathcal{S}^*$. Thus, if $A \in \mathcal{S}^*$, it follows that $\mathcal{P}(A) = 0$ or $\mathcal{P}(A) = 1$ and if η is a random variable measurable on the sample space $\mathcal{S}^*$, it follows that there is a constant c such that $\mathcal{P}(\eta = c) = 1$.

(J. Mogyoródi)

510

RUBIN, H. & TUCKER, H. G. (Oregon Univ., Portland and Univ. California, Riverside) **10.2** (10.1)
Estimating the parameters of a differential process—*In English*
Ann. Math. Statist. (1959) **30**, 641-658 (2 refs.)

Let X denote a differential process, that is to say a stochastic process such that the increments of the process in any time interval have a distribution that depends only on the length of the interval, and not otherwise on time or on changes in any other time interval. The Poisson process is a simple example. It is known that the probability distribution of such a differential process is a convolution of a normal distribution and a possible infinite number of Poisson laws. The distribution of $X(t)$ is infinitely divisible and hence the logarithm of the characteristic function can be written as

$$\log f(u) = i\gamma tu + t\int_{-\infty}^{+\infty} \{e^{iux} - 1 - iux/(1+x^2)\}(1+x^2)/x^2 dG(x).$$

Here γ is a trend term while the saltus of G at 0, i.e. $G(+0)-G(-0)$, is the variance of the normal component. Furthermore, the jumps of G (which is non-decreasing) give information as to the Poisson components.

This paper gives two estimates of this characterising function G: the first of these require the observation of the sample function at any finite number of time points. The second requires in addition that it also be possible to observe every jump of the sample function over a finite interval. In this case it is also possible to estimate the variance of the normal component. The first estimate may be biased while the second estimate is unbiased over all intervals which do not contain zero.

Finally an unbiased estimate is given for the trend term γ: this is based upon complete observation of the process for the interval [0, 1].

(D. G. Chapman)

511

SAKINO, S. & HAYASHI, C. (Institute Statistics Mathematics, Tokyo) **10.9** (11.7)
On the analysis of epidemic model I (Theoretical approach)—*In English*
Ann. Inst. Statist. Math., Tokyo (1959) **10**, 261-275 (7 refs., 6 figs.)

This paper considers the construction of epidemic models in connection with communities when the assumption of homogenous mixing is satisfied. The chance of contact is assumed to be the same for any pair of one infective and one susceptible member but not so the chance of attack for all contact members. A further simplifying assumption is that the latent period be reduced to zero by comparison with the infection period.

In section 2 and 3 of the paper a stochastic epidemic model is constructed and examined numerically by Monte Carlo methods. A diagram is given summarising the relationship between the three elements of the population at risk: susceptibles, infectives, and removals, and probabilities connecting these elements in the time interval $(t, t+\Delta t)$.

In section 4 of the paper the stochastic model is enlarged by introducing the concept of the conditional probability by which a person infected at time τ will recover in the time interval $(t, t+\Delta t)$. This overcomes the disadvantage of the model outlined in section 2 where the recovery of an infected person is independent of the infected time.

The paper concludes with a reconsideration of some points in the classical (deterministic) treatment of these problems. A further paper is envisaged which will examine the validity of the stochastic models developed in this paper using data from influenza epidemics.

(M. Siotani)

512

SCHMETTERER, L. (Inst. für Versich.-math. und Math. Statist., Hamburg) **10.5** (5.0)
Report on the literature on information theory—*In German*
Blä. Dtsch. Ges. Versich.-math. (1960) **4**, 259-266 (76 refs.)

The object of this report with its extensive bibliography is to provide a survey of the various developments that can be traced in information theory. Such a survey is also important on account of the fact that information theory plays a role in the theory of communication as well as in mathematical statistics, and the designations are not always used in the same way. A certain amount of mathematical knowledge is, however, required in order to read the report.

After a brief discussion of the fundamental contributions of Shannon and the enumeration of several works dealing with the technological aspects of the problem, the report is concerned in detail with the mathematical difficulties of information theory. Some of Shannon's concepts had to be formulated more precisely or even modified, in order to make it possible to give a correct proof of his theorems. The extension of information theory to stochastic processes (Gelfund, Jaglom) and various other generalisations are also discussed.

Finally the application of information theory in mathematical statistics is treated.

(K. Prachar)

STEUTEL, F. W. (Mathematical Centre, Amsterdam) **10.4** (2.5)
Reduction of the mean waiting time by priority assignment—*In Dutch*
Statist. Neerlandica (1959) **13**, 503-513

This paper deals with the reduction of the mean waiting time by priority assignment in a one-counter waiting time process.

In the first part, the situation is considered in which customers of two kinds arrive in independent Poisson streams, the service time of all customers of the same kind having the same distribution. Priority is given to customers of one kind. The mean waiting time averaged over all customers is obtained with use of Cobham's formulae [*J. Operat. Res.* (1954) **2**, 70-76] and compared with the mean waiting time in the case where no priorities are given. It is proved that, in order to reduce the mean waiting time, priority should be given to the customers having the smaller mean service time.

In the second part of this paper a method is used which was introduced by Prins for the special case of exponentially distributed service times: customers arrive in a Poisson stream, the service times of all customers having the same distribution with mean μ. On the assumption that the service time of an arriving customer can be estimated with sufficient accuracy the customers are divided into those having service times $\leqq c\mu$ and those with service times $>c\mu$, c being a constant. In this way a situation is created identical to the one described in the first part of this paper. Priority is given to the customers with the smaller service times. The averaged mean waiting time is calculated and (on the assumption of differentiability of the service time distribution) minimised with respect to c.

Some examples and graphs are added.

(F. W. Steutel)

STOLLER, D. S. & STOLLER, L. C. (Rand Corp. and Space Tech. Labs., Los Angeles) **10.2** (11.1)
Calculating the coefficients of certain linear predictors—*In English*
Math. Tab., Wash. (1959) **13**, 122-124 and 129 (3 refs.)

This paper presents a method of obtaining the coefficients, in a form suitable for computation, for a problem in prediction. It is assumed that observations $(x_0, x_1, \ldots, x_n)$ have been made at $n+1$ equally spaced points. It is desired to find an estimator of the next observation, y_{n+1}, under the following assumptions:

(i) the expected value of the ith observation is a polynomial of degree k.
(ii) the expected value of a linear predictor

$$(y_{n+1} = \sum_{i=0}^{n} a_i x_i)$$

is equal to the expected value of the next observation for an arbitrary polynomial of degree k.
(iii) the observations are assumed to be independent and equally variable.

The expected value of the ith observation is expressed as a linear function of orthogonal polynomials. The coefficients of the linear predictor (a_i) are determined so as to minimise the variance of the predicted value, y_{n+1}. Values of the a_i have been computed for $k = 1(1)10$ and for various values of n. They are available from the *Unpublished Mathematical Tables Repository* of MTAC. The sum of the coefficients should equal unity. This is checked to six significant figures for degrees 1 through to 8 and to five significant figures for degrees 9 and 10.

(D. Teichroew)

STREIBEL, C. T. (Lockheed Aircraft Corp., Palo Alto, California) **10.2** (10.3)
Densities for stochastic processes—*In English*
Ann. Math. Statist. (1959) **30**, 559-567 (7 refs.)

Let $x_\theta(t)$, $\theta \in \Omega$ be a parametric family of stochastic processes defined by their finite dimensional distributions: that is $F_\theta[x(t_1), \ldots, x(t_n)]$; $\theta \in \Omega$ is a given n-dimensional distribution for every finite set $\{(t_1, t_2, \ldots, t_n)\}$.

One procedure for solving the statistical problems concerning θ (for example estimation of θ) is to solve the problem for the finite dimensional family. If the solution so found has a limit in some sense for some suitably chosen limiting operation of the sets $\{t_1, t_2, \ldots, t_n\}$ and is independent of the defining set, then the statistical problem is solved for the stochastic process.

An alternative procedure was proposed by Grenander [*Ark. Mat.* (1950), **1**, 195-277]. He introduces the likelihood ratio of two processes P and Q restricted to a finite number of time points, shows that it converges to a limit as the number of points goes to infinity and this limit is the density of P with respect to Q if it exists. If this density exists, and if Q can be suitably chosen, standard statistical techniques can be applied.

The present paper gives a criterion for the existence of a density which proves convenient in several applications. A condition is also given in which a density computed for a countable number of time points is valid for the continuous parameter process.

The results are applied to a normal process with known continuous covariance function and unknown mean of the form $\sum_{i=1}^{\infty} k_i \phi_i(t)$, ϕ_i known, k_i unknown. The process is absolutely continuous with respect to the process with mean zero and the same covariance. Thus a relative density of the general process is derived and minimum variance unbiased estimates found for the k_i. The same method is used to derive the maximum likelihood estimate of the correlation parameter of the Ornstein-Uhlenbeck process.

(D. G. Chapman)

TAGA, Y. & ISII, K. (Institute of Statistical Mathematics, Tokyo) **10.5** (10.2)

On a stochastic model concerning the pattern of communication—Diffusion of news in a social group—*In English*

Ann. Inst. Statist. Math., Tokyo (1959, **11**, 25-43 (6 refs., 3 figs.)

A stochastic model is treated in which certain information I is communicated from the information source to persons, or from a person to another in a system consisting of a social group π and an information source S.

The social group π consists of M persons, and the information source S has as a a function the constant transmitting of certain information I to persons in π during the period of observation. It is assumed that none of the persons in π have the information I at time $t = 0$, and I, starting from S, is communicated from one person to another in the course of time. In the time interval $(t, t+\Delta t)$ for sufficiently small Δt, the transmission of I from a person to another has the probability $\lambda \cdot \Delta t + o(\Delta t)$, while the transmission from S to a person has the probability $\mu \cdot \Delta t + o(\Delta t)$, and the probability of two or more transmissions is $o(\Delta t)$.

It is assumed, moreover, that all possible communications in the group π are made mutually independently, and that the events which occur in $(t, t+\Delta t)$ are independent of the past history of the system up to time t. In order to characterise the diagrammatic pattern of the tree produced in the system up to time t, the authors introduce the random variables $N(t)$, $K(t)$ and $L(t)$ which represent respectively the number of persons having received I up to time t, the number of persons who have received I but not yet communicated it to any other person up to time t, and the number of persons having received I directly from S up to time t. Then it is proved that the conditional probabilities

$$p_{n,k} = \Pr\{K(t) = k \mid N(t) = n\},$$
$$p^*_{n,l} = \Pr\{L(t) = l \mid N(t) = n\}$$

are independent of time t and size M of the group π. Further, the conditional expectations

$$E_n(K) = E\{K(t) \mid N(t) = n\},\ E^*_n(L) = E\{L(t) \mid N(t) = n\},$$

and the conditional variances $V_n(K)$ of $K(t)$ and $V^*_n(L)$ of $L(t)$, given $N(t) = n$, are calculated. These results make it possible to test the assumption of homogeneity of communication in group π, or to estimate the parameter $\delta = \mu/\lambda$ which represents the ratio of intensities of two communications, namely, from S to a person and from one person to another.

It is also proved that normal approximation is valid for the conditional distributions $\{p_{n,k}\}$ of $K(t)$, $\{p^*_{n,l}\}$ of $L(t)$, and that the bi-variate statistic $(N(t), L(t))$ is a sufficient statistic for the family of distributions $\Pr\{U(t);\ \lambda,\ \mu\}$ with the parameters (λ, μ).

Further, it is noticed that among various information sources S_i the most effective one is obtained by comparing the estimates of the parameters $\delta_i = \mu_i/\lambda$ corresponding to S_i, given the same information in the social groups.

517 (Y. Taga)

URBANIK, K. (Math. Inst., Polish Academy of Sciences, Warsaw) **10.0** (10.1)

An effective example of a Gaussian function—*In Polish*

Bull. Acad. Polon. Sci. III (1959) **7**, 343-349

Let E be a Lebesgue measurable subset of the positive half-line. The set E is said to be relatively measurable, if the limit

$$|E|_R = \lim_{T\to\infty} 1/T\,|E\cap\{t: 0\leqq t\leqq T\}|$$

exists (by $|Z|$ we denote the Lebesgue measure of Z). For every interval $I = \{t: a\leqq t<b\}(0\leqq a<b)$ and for every real-valued function $f(t)$ defined on the positive half-line we shall use the following notation:

$$f^*(\mathcal{I}) = f(b)-f(a),\ I+t = \{u+t: u\epsilon I\}.$$

We say that a continuous function $f(t)$ defined on the positive half-line is a Gaussian function if for every system of disjoint intervals $I_1, I_2, \ldots, I_k$ and for every system of real numbers $x_1, x_2, \ldots, x_k$ the set $\bigcap_{j=1}^{k} \{t: f^*(I_j+t)<x_j\}$ is relatively measurable and

$$\left|\bigcap_{j=1}^{k}\{t: f^*(I_j+t)<x_j\}\right|_R = \prod_{j=1}^{k}\phi\left(\frac{x_j}{\sqrt{|\mathcal{I}_j|}}\right),$$

where $$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2}\,du.$$

It is well known that almost all realisations of the normalised Brownian movement processes are Gaussian functions. In the note an effective example of a Gaussian function is given.

(L. Kubik)

WINSTEN, C. B. (Imperial College, London) **10.4** (10.1)

Geometric distributions in the theory of queues—*In English*

J. R. Statist. Soc. B (1959) **21**, 1-22 (8 refs., 2 tables, 2 figs.)

A new explanation in probability terms is given for the occurrence of geometric or conditional geometric distributions in some queueing problems. Suppose customers arrive at regular intervals, and that between each pair of arrivals the *potential* number of customers served is a random variable which is distributed independently and identically for separate intervals. The *actual* number served is equal to this potential number if the queue is long enough; otherwise the queue-length is reduced to zero before the next arrival. This formulation of the queueing situation is equivalent to specifying an infinite transition matrix in which, except for the first column, the elements in each of the diagonals are equal; also there are zeros everywhere above the first super-diagonal.

Now the probability that a queue-trajectory (i.e. history of the queue-length for a single realisation of the process) which passes through the value q at time zero, passes through q again at time t without having fallen below q in between, is independent of q. Assuming equilibrium, the probability that any trajectory passes through q at time zero can be calculated as a sum of terms relating to the last time the trajectory passed through $q-1$; and this relation is independent of q. Hence it follows that the equilibrium distribution of queue-length must be geometric.

The argument applies in other cases, for example if there is more than one server. The transition matrix is as above except that now more than one initial column may be arbitrary. The queue-length is geometric, except for the initial terms.

Next it is assumed that customers arrive late, the amounts of lateness being independent and identically distributed and being bounded by unity, so that customers cannot arrive in the wrong order; service-times are now exponential. The limiting queue-length distribution is geometric, except for the first term. The waiting-time is exponential with a lump at zero; the mean waiting-time is a minimum when the arrivals are equally spaced, and maximum when the lateness distribution approaches a certain binomial distribution on zero and unity.

The case when the lateness extends up to two time-units is investigated.

Finally, if the queue-length can increase by at most k at each arrival-epoch, the equilibrium queue-length probabilities satisfy a kth order difference equation. This covers the case of k-stage (Erlang) service.

(C. L. Mallows)

Editorial Note: see abstract No. 522, **10.4.** Winsten

519

WINSTEN, C. B. (Report on a Discussion on paper by) **10.4** (10.1)

Geometric distributions in the theory of queues—*In English*

J. R. Statist. Soc. B (1959) **21**, 23-35.

The contributors to the discussion were Lindley, Wishart, Foster, Takács, Kiefer, Lewis, Jackson, Conolly (in writing), and Mallows: the author replied.

Lindley pointed out that the usual assumption of independent inter-arrival times could arise naturally when the input to the queueing system was the output from a previous system, and that Winsten's method could be applied to this case, at least when the service-times were exponential. The usual technique for studying a stochastic process was to obtain a relation between a state-probability at time zero and the probabilities of all the possible states at some previous fixed time. Winsten's paper demonstrated that other types of relation may be valuable, in terms of the probabilities of trajectories (of all possible lengths) through a fixed state. In many situations, for example in the finite dam problem, this change of emphasis may lead to new results.

Wishart noted an extension of the argument to the system $GI/E_k/s$, and described a Monte Carlo experiment relating to equidistant scheduled arrivals, exponential lateness distribution, and single exponential service.

Foster remarked that the methods of the paper applied also when customers could be early as well as late. For the system $GI/E_k/1$, one can imagine customers arriving in batches of k; with exponential service; there is an integral relation between the queue-length generating functions for this model and for the model obtained by counting only every kth customer. A similar approach can be used to solve the system $E_k/G/1$.

Takács presented a more general formulation of the queueing process, which leads naturally to a proof of the geometric results in the cases considered by Winsten, and pointed out an analogy between the $M/GI/1$ and the $GI/M/s$ (with finite waiting-room) processes.

Keifer mentioned the problem of simultaneous queues, with customers joining the shorter queue; and of customer impatience, where each customer leaves without being served if he has to wait for longer than a specified time.

Lewis said that if the lateness distribution has a large variance, certain sums arising in the specification of the properties of the arrival process could be replaced by integrals, and simple results obtained.

Jackson distinguished three classes of lateness distribution, namely, everybody early; everybody late; neither late nor early. Conolly also referred to the problem of early arrivals. Mallows showed that Winsten's method applied also when customers may baulk.

The author replied, and discussed the system $M/M/s$, showing the effect of customer impatience; he noted that a point raised by Mallows, the analogy given by Takács, and some recent results on reservoirs, all follows from symmetries in the corresponding queue-trajectories.

(C. L. Mallows)

Editorial Note: see abstract No. 521, **10.4.** Winsten

520

WEINSTEIN, A. S. (New York State Department of Mental Hygiene) **10.7** (11.1)
Alternative definitions of the serial correlation coefficient in short autoregressive sequences—*In English*
J. Amer. Statist. Ass. (1958) **53**, 881-892 (28 refs., 4 tables)

This review paper summarises alternative definitions of the serial coefficient and alternative methods of estimation are investigated with the aid of series of known autoregressive properties. Included in the investigation are the ordinary circular and non-circular definitions as well as some approximations. The bias introduced by the necessity of correcting for an unknown mean is considered. The series used in the estimation are constructed to represent a time series of monthly observations and estimates based upon the segmentation of the series into observations for a given month presented. The methods employed are explained in greater detail in a previous paper by the same author given in the references.

(W. A. Fuller)

ZUBRZYCKI, S. (Math. Inst., Polish Academy of Sciences, Warsaw) **10.5** (2.0)
Concerning Yule's characteristic of style—*In English*
Zastosowania Mat. (1959) **4**, 328-331

This note is concerned with a discussion of two characteristics of style used in statistical linguistics and based on counts of occurrences of particular words in a text. The first of them is due to Yule [*Statistical Study of Vocabulary* (1944), Cambridge], the other to Herdan [*Zeit. angew. Math. Phys.* (1955) **6**, 332-334]. In this note, a probabilistic explanation is given for that these characteristics are reasonably independent of the sample size. The difference between them is also stressed. The explanation given differs from those contained in Herdan's earlier publications [see also *Zeit. angew. Math. Mech.* (1956) **36**, 72 and his book *Language as Choice and Chance* (1956), Groningen]. The main argument is based on a relation which is also contained in Herdan's later paper [*J. Appl. Math. Phys.* (1958) **9**, 69-73].

(S. Zubrzycki)

BELLMAN, R. & DREYFUS, S. (RAND Corp., Santa Monica, California) **11.5** (0.8)
Functional approximations and dynamic programming—*In English*
Math. Tab., Wash. (1959) **13**, 247-251 (5 refs.)

One of the difficulties which arise in the numerical solution of problems is that as the number of dimensions of the problem increases, the number of calculations which must be performed increases as a power. For example, if a function must be evaluated at m points in one dimension, it must be evaluated at m^2 points in two dimensions, m^3 in three dimensions, etc. The authors suggest that instead of computing a function value for each grid point the functions be represented by a set of orthogonal polynomials; for example, they might be Legendre polynomials with the coefficients chosen to be the Christoffel numbers.

The following table gives a comparison of the number of values which must be stored if a grid of size Δ is used as compared with using orthogonal functions of degree R, and d dimensions, in which case $(R+1)(R+2)\ldots(R+d)/d$ coefficients would be required. Using $\Delta = 0{\cdot}01$ and $R = 5$, 10, the number of values which must be stored is as follows:

Dimension (d)	$(1/\Delta)^N$	$R = 5$	$R = 10$
1	10^2	6	11
2	10^4	21	66
3	10^6	56	286
4	10^8	126	1001
5	10^{10}	252	3003

It can be seen that the direct method would require a storage capacity completely beyond that of present machines whereas the method of approximating by a series of orthogonal functions is now practicable.

The authors apply this technique to a problem in dynamic programming; the problem is to determine $\{v_n\}$ so as to minimise the function

$$F_N = \sum_{k=1}^{N} u_k^2 + \lambda \sum_{k=1}^{N} v_k^2$$

where (*a*) $u_{n+1} = 2u_n - u_n^2 + v_n$, $u_0 = c$,
(*b*) v_n must be chosen so as to keep u_{n+1} within the interval $[-1, 1]$.

Let $f_N(c)$ be the value of F_N obtained when the minimum values of the $\{v_n\}$ are used.

$$f_N(c) = \text{Min}\,[c^2 + \lambda v^2 + f_{N-1}(2c - c^2 + v)]$$

for $N \geqq 2$ and $f_1(c) = c^2$.

The following table shows the agreement between the grid technique and the orthogonal polynomial technique for the case $N = 6$; $f_2(c)$ is determined by using the grid and $f_6^*(c)$ by using the Legendre polynomials.

c	$f_6^*(c)$	$f_2(c)$
1·0	1·782	1·77
0·8	1·370	1·36
0·2	0·153	0·145
0·0	0·006	0·0
−0·2	0·202	0·20
−0·8	4·876	4·89
−1·0	8·666	8·67

The agreement is regarded as quite good.

523

(D. Teichroew)

GREEN, B. F. Jr., SMITH, J. E. K. & KLEM, L. (Massachusetts Inst. Tech., Lexington) **11.5** (2.6)
Empirical tests of an additive random number generator—*In English*
J. Ass. Comp. Mach. (1959) **6**, 527-537 (8 refs., 8 tables, 1 fig.)

Most of the presently used procedures for generating random numbers on digital computers are based on the multiplicative process; e.g.,

$$X_j = aX_{j-1} (\text{mod } 2^r).$$

In this process one random number is obtained from the previous one by one multiplication. This procedure has been studied extensively and has been found satisfactory. However, it may be possible to reduce the required computing time by using, instead of a multiplication, an addition, e.g.,

$$X_j = (X_{j-1} + X_{j-n}) \text{ mod } 2^r.$$

In this method, the storage is required for n numbers but each new random number is obtained by a single addition. This usually represents a decrease in computing time; the exact amount will depend on the particular machine that is used.

This paper discusses the additive method. The length of the period will depend on n, r and the initial set of n numbers that are used. The period is sufficiently large for practical purposes. A number of tests were run on numbers generated by this process to determine their value as random numbers. The tests included the frequency test, the "poker" test, serial correlation test and the run test. The process passes the frequency, poker and serial correlation tests but fails the run test for n less than 16. It passes the run test for $n = 16$ and presumably for $n > 16$. If alternate numbers are discarded then the additive process passes the run test for $n = 6$ and presumably for larger n. The tests were conducted for $r = 15$ and 35 using the Lincoln Laboratory's Memory Test computer and for $r = 35$ using an IBM 704. The authors conclude that this procedure can be used, provided the deficiency on the run test is taken into account either by dropping alternate numbers or by using $n = 16$.

(D. Teichroew)

HEALY, M. J. (Rothamsted Exp. Station, England) **11.5** (9.0)

The analysis of field experiments on an electronic computer—*In English Biom. Zeit.* (1959) **1**, 210-214 (6 refs.)

The paper deals with some experiences obtained in the analysis of field experiments by use of the Elliot/NRDC 401 computer at Rothamsted. The computer is of medium size and has a storage capacity of 2944 words (of 32 " bits " each). Input and output are on 5-hole punched paper tape. The first part of this paper treats the input, by means of a simple example and the author shows how to proceed in the case of a larger amount of input data and how to use the computer for preliminary computations.

In the second part (Analysis) some programmes in use at Rothamsted are mentioned (Randomised Blocks, Latin Squares, and those for factorial experiments). To avoid " overflow ", the author recommends the use of a working mean. In the case of a 2^n factorial experiment, Hartley's method of working with deviations from expected values turned out to be very useful. A valuable check consists in the computation of residuals and the residual sum of squares.

The third part describes the output. The use of headings and the description of the analysed variables for the layout of the output data are recommended. At the end of the paper some special features are discussed. In the case of missing values, an efficient iterative method due to Healy and Westmacott is mentioned.

(J. Roppert)

525

MORTARA, G. (University of Rome) **11.8** (2.1)

Intensity index for international trade between two countries—*In Italian Industria* (1959) 17-26 (1 ref., 3 tables)

In this article the author exposes the methodological problem of the calculation of an index of the relative intensity of the trade between two countries in relation to their world-trade.

He presents many kinds of indexes calculated among the European Common Market countries.

These indexes are ratios of the values of the trade between two countries to the values of the trade between the considered country and the other countries of the world, the value of the trade having been calculated with reference to the population. The above indexes are called unilateral indexes by the author and the reference to the population can be replaced by reference to other factors, e.g. national income or territory.

With this procedure we have two comparative indexes one of them measuring the intensity of the trade of a country with reference to the trade of this country with all other considered countries. It is also possible to calculate bilateral indexes as the average of two unilateral indexes. The calculation of a bilateral index is made with the geometric mean because the unilateral indexes are ratios.

Referring to bilateral indexes the author says that it is very difficult to identify them with a definite economic meaning.

(P. Bandettini)

526

MOSHMAN, J. (Corporation for Economic and Industr. Res., Arlington, Va.) **11.7** (4.7)
The application of sequential estimation to computer simulation and Monte Carlo procedures—*In English*
J. Ass. Comp. Mach. (1958) **5**, 343-352 (10 refs., 1 fig.)

This paper describes some of the sequential estimation techniques that could be used to terminate sampling when electronic computers are used for random sampling. In the past, very large samples have usually been required in order to obtain estimates with reasonably small variances. A logical method of reducing average sample size is to consider a technique of sequential estimation to provide for continual machine sampling until a sufficiently large sample is accumulated so that one obtains an estimate with a predetermined variance, coefficient of variation or confidence interval.

The methods presented here have all been used in the literature. They include the following:

(i) Stein's two-stage sampling technique [*Ann. Math. Statist.* (1945) **16**, 243-258] can be used to estimate the mean of a normal distribution having unknown variance; for a given initial sample of size N_0, a second sample of size $N-N_0$ is obtained so that the confidence coefficient is at least $1-\lambda$ (λ being specified in advance). The present author has given an objective rule for choosing N_0 [*Ann. Math. Statist.* (1958) **29**, 1271-1275]. Anscombe [*J. R. Statist. Soc.* B (1953) **15**, 1-21] suggested a second-order asymptotic sequential procedure.

(ii) Girshick, Mosteller and Savage [*Ann. Math. Statist.* (1946) **17**, 13-23] reported on a sequential procedure for estimating the parameter p of a binomial population. Stockman and Armitage [*Suppl. J. R. Statist. Soc.* (1946) **8**, 104-112] have provided the means of using this procedure, which are feasible in, and only in, the framework of a computer for most practical problems.

(iii) Anscombe [*Biometrika* (1949) **36**, 455-458] suggests a procedure for constructing a boundary for a sequential process in which the mean θ, which has a variance $V(\theta)$ which is a finite function of θ, is estimated with fixed variance or coefficient of variation.

An example is given to compare the Stein two-sample procedure with Anscombe's sequential procedure.

(D. Teichroew)

Editorial Note: see abstract No. 105, **8.3.** Moshman

MOTOO, M. (Institute of Statistical Mathematics, Tokyo) **11.7** (10.1)
Some evaluations for continuous Monte Carlo method by using Brownian hitting motions—*In English*
Ann. Inst. Statist. Math., Tokyo (1959), **11**, 49-54 (3 refs.)

This paper treats the so-called δ-truncated spherical process [see also: Muller, *Ann. Math. Statist.* (1956) **27**]. By this process, Muller has constructed an estimator of the solutions to Dirichlet's problem.

The main part of this paper is devoted to the evaluation of the upper bound of the mean square error of the estimator and that of the mean number of steps needed for this process. Assuming the boundary and the boundary function of Dirichlet's problem are sufficiently smooth, these mean values can be replaced by the mean sojourn times of certain Brownian hitting motions which are given by a random changing of the time scale from Brownian motion. By evaluating these mean sojourn times, which are solutions of certain differential equations, the author obtains the desired upper bounds.

(H. Akaike)

PANIZZON, G. (Institute of Statistics, University of Padua) **11.0** (–.–)

Demand elasticity of some foodstuffs according to Italian experience—*In Italian*

Riv. Int. Sci. Econ. Comm. (1959) **6**, 1148-1177 (26 refs., 10 figs.)

The methodological interest of this article consists in the method used for the construction of the dynamic demand curve, based on market statistics.

Budget equation $\Sigma p_{t,i}\, q_{t,i} = a_t$ ($p_{t,i}$ = prices; $q_{t,i}$ = quantities demanded; a_t = income; t = time) is taken as a starting point. It is noted that this equation would be " solvable " if all prices and all quantities, referring to a given period of time, could be expressed in terms of one price and one quantity.

This is maintained to be feasible whenever, over relatively short periods of time, the ratios between the price of one item to that of any other can be shown to be approximately constant. According to the author, this happens to be the case in Italy for the nine items considered during the periods 1891-1899 and 1950-1956. Actual prices are then substituted by new ones ($p^*_{t,i}$ = calculated prices) on the basis of a strictly constant prices ratio. On the other hand, the theory of Pareto (on the standard of living) leads the author to some expressions in the form $f_i(q_{t,1}) = q_{t,i}$ associating the quantities of the various goods to the quantity of one item (e.g., $q_{t,1}$).

An equation is obtained from which known $p_{t,i}$ and a^*_t (being the interpolated values of income) together with the unknown value of $q_{t,1}$ (the reference item) are easily drawn: and with this all other values $q_{t,i}$.

Theoretical quantities, together with the new prices already calculated constitute the dynamic demand curve. Elasticity is measured by means of Schultz's formulae [*Statistical laws of demand and supply*, (1928) Chicago, p. 12].

(O. Cucconi)

SCHWARZ, K. (Statistisches Bundesamt, Wiesbaden) **11.8** (11.0)

Index numbers in migration statistics—*In German*

Allg. Statist. Arch. (1959) **43**, 17-34 (4 refs., 3 tables)

The author distinguishes three kinds of index numbers in migration statistics:

(i) the mobility of a certain population,
(ii) migration between two regions, or " volumes of migration streams "
(iii) the effect of migration on the size and composition of a population.

In (i) he shows that the correct way to calculate rates of mobility is to divide the number of outward movements (from internal migration as well as emigration) by the number of the population " exposed to the risk of migration." If there is a considerable change in the size of the population during the period under review, the average number should be used.

In order to measure the migration stream from one region (A) to another region (B), he proposes also to calculate the rate of mobility. The number of persons moving to (B) is to be divided by the number of people living in (A). This formula may be improved by taking into account the area of B. The author favours the formula:

$$w' = \frac{M}{E.p}.1000 \qquad w' = (m/E.p)1000$$

where M = persons moving from (A) to (B)
E = population of (A)
p = area of (B) as a fraction of the total territory.

The author refers to other possibilities of measurement, which also take into account the size of population of (B). As an alternative he proposes to measure the " attractive " forces of a place (B) with respect to another place (A) by means of coefficients of association in contingency tables analogous to the concepts of social mobility.

Considering (iii), the author briefly discusses rates which are based on net migration. Finally he describes the measure of effectiveness of migration, namely the fraction of net migration and the total number of external migrations.

(K. Weichselberger)

TEDESCHI, B. (University of Rome) **11.8** (2.1)

Concerning an index for measuring the change, over a period of time, of the mortality in a community—*In Italian*

G. Ist. Ital. Attuari (1958) **21**, 13-19 (7 refs., 5 tables)

The object of this work is to fix a suitable index for measuring the change in mortality over a period of time.

Denoting by $q(x, t)$ the death rate for a person of age x, belonging to a given community, within calendar year t, the author observes that in order to compare the mortality of the persons of age x belonging to that community, within year t', with the mortality of the persons of age x of that community, within year t'', any one of the following indexes is commonly used:

$$I_1(x, t', t'') = q(x, t'')/q(x, t'); \quad I_2(x, t', t'') = q(x, t')-q(x, t'');$$
$$I_3(x, t', t'') = [q(x, t')-q(x, t'')]/q(x, t').$$

Through the comparison of the values obtained by applying the said indexes to the death rates of the male population of Italy, over different periods of time, it is observed that the two indexes I_2 and I_3 may lead to contradictory results, whereas indexes I_1 and I_2 are not comparable because they are not homogeneous. The choice of the wrong index may lead to arbitrary conclusions.

The author, referring by way of example to years 1930 and 1950 and making successive decompositions among the causes which brought about changes in mortality between the said two years, then takes into consideration a group α of the causes which did not bring about any changes and group ε of the causes which did bring about changes.

If $q_\varepsilon(x, 1930)$ is the death rate for a person of age x connected with cause ε alone in 1930, $q_\alpha(x, 1930)$ is the analogous rate connected with cause α alone and $q_{\alpha\varepsilon}(x, 1930)$ the rate supplied by 1930 statistics, depending on causes α and ε. Moreover, if in 1950 the death rate for a person of age x, $q_\alpha(x, 1950)$, depends on cause α alone the result, considering $q_\varepsilon(x, 1930)$ as a pure probability of death (according to Karup), is

$$q_\varepsilon(x, 1930) = \frac{q_{\alpha\varepsilon}(x, 1930)-q_\alpha(x, 1950)}{1-q_\alpha(x, 1950)}.$$

The index suggested by the author for the measurement of the change in mortality from 1930 to 1950 is $I(x, 1930, 1950) = q_\varepsilon(x, 1930)$, $q_\varepsilon(x, 1930)$ being given by the foregoing expression. Such an index can obviously be applied with reference to two years whatever t' and t'', replacing 1930 and 1950 by t' and t''.

The values of this new index and of index I_2, as defined above, resulted very similar for the ages corresponding with low death rates; whereas the higher the death rate $q_\alpha(x, 1950)$, the greater the difference between them.

(A. Del Chisco)

531

WARD, D. H. (National Coal Board, Manchester, England) **11.3** (6.2)

A nomogram for calculating partial correlation coefficients—*In English*

Inc. Statist. (1959) **9**, 139-141 (1 ref.)

Presenting an alternative graphical method to that of Liddell [*Inc. Statist.* (1957) **7**, 167-169], the author comments that although the indiscriminate use of the correlation coefficient is now out of fashion it is still of value to have a rapid method of approximate calculation for the partial correlation coefficient. The significance testing of sample (partial) regression coefficients is a particular example.

The nomogram presented in this short paper deals with the graphical calculation of a partial correlation coefficient from three total correlation coefficients. By using a straightforward method involving determinants, the author obtains an expression for the following:

$$(1, rp),\ (0, r) \text{ and } \left(\frac{\sin\theta\sin\phi}{\sin\theta\sin\phi-1}, \frac{\cos\theta\cos\phi}{\sin\theta\sin\theta-1}\right)$$

where $rp = r_{12.3} = r\ \mathrm{cosec}\,\theta\ \mathrm{cosec}\,\phi-\cot\theta\cot\phi$

$r = r_{12}$

and $\cos\theta = r_{13}$

$\cos\phi = r_{23}$

These points are collinear and enable the nomogram to be constructed.

(W. R. Buckland)

532